

The nonparametric bootstrap for the current status model

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Abstract: It has been proved that direct bootstrapping of the nonparametric maximum likelihood estimator (MLE) of the distribution function in the current status model leads to inconsistent confidence intervals. We show that bootstrapping of functionals of the MLE can however be used to produce valid intervals. To this end, we prove that the bootstrapped MLE converges at the right rate in the L_p -distance. We also discuss applications of this result to the current status regression model.

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1. Introduction

In the current status model, the variable of interest is a survival variable X with distribution function F_0 . However, instead of observing the exact survival time X , a censoring variable $T \sim G$ is observed together with the indicator $\Delta = 1_{X \leq T}$. Such data arise naturally in clinical trials when a patient can only be checked at one measurement due to destructive testing. A lot of research has been published on the behavior of the maximum likelihood estimator (MLE) F_n of the distribution function F_0 . The limiting distribution of $n^{1/3}(F_n(t_0) - F_0(t_0))$ is after scaling by the constant $\kappa = \{4F_0(t_0)(1 - F_0(t_0))f_0(t_0)/g(t_0)\}^{1/3}$ given by,

$$\mathbb{C} = \arg \max_t \{W(t) - t\},$$

where W is a two-sided Brownian motion with $W(0) = 0$ (see [16]). Other estimators with similar asymptotic properties are Chernoff's estimator of the mode ([4]), the Grenander estimator ([8]) of a nonincreasing density, Manski's maximum score estimator ([22]) and Rouseeuw's least median of squares estimator ([24]). A general framework for cube-root n asymptotics is given in [20].

In this paper we investigate the behavior of Efron's nonparametric bootstrap method ([7]) for constructing confidence intervals for smooth functionals of the

MLE. It is known that the nonparametric bootstrap is inconsistent for generating the limit distribution of the MLE. The authors of [2] prove that (conditional on the data),

$$n^{1/3}\{4F_0(t_0)(1 - F_0(t_0))f_0(t_0)/g(t_0)\}^{-1/3}\{\hat{F}_n(t) - F_n(t)\} \\ \xrightarrow{\mathcal{D}} \arg \max_t (W(t) - t^2) - \arg \max_t (W(t) + \hat{W}(t) - t^2),$$

where \hat{F}_n is the bootstrap MLE and W and \hat{W} are two independent two-sided Brownian motions originating at zero. A similar result is obtained in [21] for the Grenander estimator. Constructing asymptotic confidence intervals based on Chernoff's distribution and the normalizing constant κ is complicated by the need to compute the critical values of \mathbb{C} and to estimate the density f_0 consistently. Since this turns out to be a rather difficult task several alternative bootstrap methods have been proposed based on resampling from a smooth estimate. [26] consider a smooth kernel estimate \tilde{F} of F_0 and resample the Δ_i from a Bernoulli distribution with probability $\tilde{F}(T_i)$, while keeping the censoring variables T_i fixed. [21] proposed a similar resampling scheme for the Grenander estimator. Both methods result in consistent estimation of the (suitably standardized) distribution \mathbb{C} conditional on the original data. A drawback of this approach is that smoothness conditions of F_0 are used which allow faster than cube-root n estimation of F_0 . This raises the question if one should really use confidence intervals based on the MLE instead of on a faster converging estimate.

This latter procedure is followed in [12], where the authors consider constructing confidence intervals around the smoothed maximum likelihood estimator (SMLE) of F_0 in the current status model. The SMLE is a kernel estimate based on the MLE with an asymptotic normal, instead of Chernoff's limiting distribution ([14]). The bootstrap method proposed in [12] is however still based on the smooth bootstrap procedure described in [26] and not on Efron's nonparametric bootstrap. We show in this paper that the construction of confidence intervals around the SMLE based on the nonparametric bootstrap can also be proved to be valid. Simulation studies in [15] already indicated this result, but the proof that the nonparametric bootstrap works for our functionals was still missing. An important difference with the smooth bootstrap in [13] is that for the centering of the estimates in the nonparametric bootstrap samples the SMLE of the original sample is used, whereas this will not work for the resampling as proposed in [13]; in the latter case one needs to center the estimates in the bootstrap samples by a kernel convolution of the SMLE in the original sample.

Although it is argued in [6] that the naive bootstrap will not work for their goodness-of-fit test for monotone functions, based on the Grenander estimator, no theoretical justification for this conjecture is given. Other examples where a smooth bootstrap procedure is used, are the likelihood ratio type two-sample test for current status data proposed by [9] and the test for equality of functions under monotonicity constraints proposed by [5]. Both tests establish asymptotic normality for the test statistic considered.

The paper is organized as follows: In Section 2 we introduce the current status model and review some interesting properties of the MLE. The validity of the nonparametric bootstrap is discussed in Section 3. In Section 4 we provide two examples to illustrate the applicability of our result. In the first example we construct pointwise confidence intervals based on the smoothed MLE in the current status model. The second example deals with doing inferences for a finite dimensional regression parameter in the current status linear regression model. For both examples, the theoretical and finite sample behavior of the nonparametric bootstrap is discussed. Section 5 presents some concluding remarks. The proofs of our results are given in Section 6.

2. The current status model and the MLE

Let $Z_1 = (T_1, \Delta_1), \dots, Z_n = (T_n, \Delta_n)$ be an i.i.d. sample from the probability space $([0, R] \times \{0, 1\}, \mathcal{A}, P)$, where $\Delta_i = 1_{X_i \leq T_i}$ and $R > 0$. The X_i are interpreted as (nonnegative) survival times with distribution function F_0 . Instead of observing X , a censoring variable $T \sim G$ is observed (with density g) independent of X . One could say that in the current status model, each observation Z_i represents the current status of the item i at time T_i . The density of Z_i with respect to the product of Lebesgue measure and counting measure on $[0, R] \times \{0, 1\}$ is given by,

$$p_{F_0}(t, \delta) = [\delta F_0(t) + (1 - \delta)\{1 - F_0(t)\}] g(t).$$

The maximum likelihood estimator F_n is defined as the maximizer of the log likelihood given by (up to a constant not depending on F),

$$\ell_n(F) = n^{-1} \sum_{i=1}^n [\Delta_i F(T_i) + (1 - \Delta_i)\{1 - F(T_i)\}], \quad (2.1)$$

over all distribution functions $F : [0, \infty] \mapsto [0, 1]$. [16] show that the MLE can be characterized as the left-continuous slope of the greatest convex minorant of a cumulative sum diagram consisting of the points $(0,0)$ and

$$\left(i, \sum_{j \leq i} \Delta_{(j)} \right),$$

where we let $T_{(j)}$ denote the j th order statistic of the T_i and $\Delta_{(j)}$ be the Δ_i corresponding to it (assuming no ties are present in the data). An important property of the MLE is the so-called *switch relation*, see [15] p. 69. Let G_n be the empirical distribution function of T_1, \dots, T_n and define the process V_n by:

$$V_n(t) = n^{-1} \sum_{i=1}^n \Delta_i 1_{\{T_i \leq t\}}, \quad (2.2)$$

and the process (in a) U_n by:

$$U_n(a) = \operatorname{argmin}\{t \in \mathbb{R} : V_n(t) - aG_n(t)\}. \quad (2.3)$$

Then, taking $a = F_0(t_0)$, we get the *switch relation*:

$$F_n(t) \geq a \iff U_n(a) \leq t,$$

see also Figure 1.

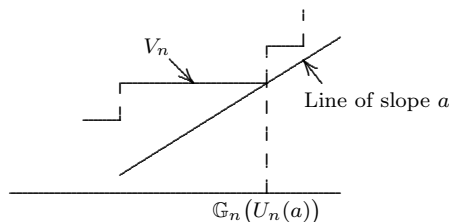


Fig 1: The switch relation.

3. Bootstrapping the MLE

In this section we establish properties of the bootstrap MLE \hat{F}_n based on the nonparametric bootstrap proposed by [7]. Our main concern is to show that conditional on the data Z_1, \dots, Z_n , we have,

$$E \left\{ \left\| n^{1/3} \left\{ \hat{F}_n - F_0 \right\} \right\|_p \mid Z_1, \dots, Z_n \right\} = O_p(1), \quad (3.1)$$

and

$$\sup_{t \in [0, R]} E \left\{ n^{1/3} \left| \hat{F}_n(t) - F_0(t) \right| \mid Z_1, \dots, Z_n \right\} = O_p(1). \quad (3.2)$$

Denote the empirical probability measure of Z_1, \dots, Z_n by \mathbb{P}_n . The bootstrap empirical measure is

$$\hat{\mathbb{P}}_n = n^{-1} \sum_{i=1}^n M_{ni} 1_{Z_i},$$

where 1_{Z_i} denotes the points mass at $Z_i = (T_i, \Delta_i)$ and

$$M_n = (M_{n1}, \dots, M_{nn}) \sim \text{multinomial}(n, n^{-1}, \dots, n^{-1}),$$

is a vector of multinomial weights, independent of Z_1, \dots, Z_n . The bootstrap MLE \hat{F}_n is computed using the weighted cumulative sum diagram formed by

the point $(0, 0)$ and

$$\left(\sum_{j=1}^i M_{n(j)}, \sum_{j=1}^i M_{n(j)} \Delta_{(j)} \right),$$

where $M_{n(j)}$ corresponds to the multinomial weight corresponding to $T_{(j)}$. The bootstrap MLE \hat{F}_n is then calculated from the left-continuous slope of the convex minorant of this cusum diagram.

To complete notation, we suppose that the vectors $((Z_1, \dots, Z_n), M_n), n = 1, 2, \dots$ are defined on the product space $([0, R] \times \{0, 1\})^\infty \times \mathbb{Z}_+^\infty, \mathcal{B}, P_{ZM}$, where \mathbb{Z}_+ is the set of nonnegative integers and \mathcal{B} is the collection of Borel sets, generated by the finite dimensional projections. We say that a real-valued function Γ_n defined on the joint probability space is of order $o_{P_M}(1)$ in probability if for all $\epsilon, \eta > 0$:

$$P^* (P_{M|Z} \{|\Gamma_n| > \epsilon\} > \eta) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where P^* denotes outer probability and $P_{M|Z}$ is the conditional probability measure w.r.t. the weights, given the sample Z_1, \dots, Z_n .

To establish (3.1), we need the following result, which is a bootstrap version of Lemma 11.5 in [15].

Lemma 3.1. *Suppose F_0 has a continuous density f_0 with support $[0, R]$ that satisfies,*

$$0 < \inf_{t \in [0, R]} f_0(t) < \sup_{t \in [0, R]} f_0(t) < \infty.$$

Also suppose that the observation distribution G has a continuous derivative g that stays away from zero and infinity on $[0, R]$. Let

$$U(a) = F_0^{-1}(a) \quad 0 < a < 1,$$

and define the process

$$\hat{U}_n(a) = \operatorname{argmin}\{t \in [0, R] : \hat{V}_n(t) - a\hat{G}_n(t)\} \quad 0 < a < 1,$$

with processes \hat{V}_n and \hat{G}_n defined by,

$$\hat{V}_n(t) = \int_{u \in [0, t]} \delta d\hat{\mathbb{P}}_n(u, \delta) \quad \text{and} \quad \hat{G}_n(t) = \int_{u \in [0, t]} d\hat{\mathbb{P}}_n(u, \delta) \quad t \in [0, R]. \tag{3.3}$$

Then there are positive constants K_1 and K_2 , such that, for all $x > 0$ and all large n :

$$\left\{ \exists x \in [0, R] : P_{M|Z} \left\{ n^{1/3} \left| \hat{U}_n(a) - U(a) \right| \geq x \right\} > K_1 e^{-K_2 x^{3/2}} \right\} = o_p(1),$$

where $\{A\}$ denotes the indicator 1_A of the event A .

Lemma 3.1 implies that the probability that for all $x \in [0, R]$,

$$P_{M|Z} \left\{ n^{1/3} \left| \hat{U}_n(a) - U(a) \right| \geq x \right\} \leq K_1 e^{-K_2 x^{3/2}}$$

tends to 1 as $n \rightarrow \infty$. The proof of Lemma 3.1 is given in Section 6. The proof uses empirical process theory and results on tail probabilities for $\|\sqrt{n}(\hat{\mathbb{P}}_n - \mathbb{P}_n)\|_{\mathcal{F}}$ for classes \mathcal{F} with finite entropy integrals. Similar results are proved using martingale theory in Section 11.2 of [15] for the original sample and in [12] for a smooth bootstrap empirical process. Since

$$E_{M|Z} \left[n^{1/3} \{ \hat{F}_n(t) - F_0(t) \}_+ \right]^p = \int_0^\infty P_{M|Z} \left\{ n^{1/3} \{ \hat{F}_n(t) - F_0(t) \} \geq x \right\} p x^{p-1} dx,$$

where $\{ \hat{F}_n(t) - F_0(t) \}_+$ denotes the positive part of $\{ \hat{F}_n(t) - F_0(t) \}$ and since,

$$\begin{aligned} P_{M|Z} \left\{ \hat{U}_n \left(a + n^{-1/3} x \right) \leq t_0 \right\} \\ = P_{M|Z} \left[n^{1/3} \left\{ \hat{U}_n \left(a + n^{-1/3} x \right) - U \left(a + n^{-1/3} x \right) \right\} \right. \\ \left. \leq n^{1/3} \left\{ t_0 - U \left(a + n^{-1/3} x \right) \right\} \right]. \end{aligned}$$

it follows from Lemma 3.1 and the *bootstrapped switch relation* given by

$$P_{M|Z} \left\{ n^{1/3} \{ \hat{F}_n(t_0) - F_0(t_0) \} \geq x \right\} = P_{M|Z} \left\{ \hat{U}_n \left(a + n^{-1/3} x \right) \leq t_0 \right\},$$

that there exists a positive constant $K > 0$ such that,

$$\left\{ \exists t \in [0, R] : E_{M|Z} \left| \hat{F}_n(t) - F_0(t) \right|^p > K n^{-p/3} \right\} = o_p(1).$$

In particular, there exists a $K_1 > 0$ such that:

$$P \left\{ \sup_{t \in [0, R]} E_{M|Z} \left| \hat{F}_n(t) - F_0(t) \right| > K_1 n^{-1/3} \right\} \rightarrow 0, \quad n \rightarrow \infty,$$

and likewise there exists a $K_2 > 0$ such that:

$$P \left\{ E_{M|Z} \left\| \hat{F}_n - F_0 \right\|_2 > K_2 n^{-1/3} \right\} \rightarrow 0, \quad n \rightarrow \infty.$$

4. Applications

In this section we show how (3.1) can be used to justify the bootstrap validity for drawing inferences in models which can be estimated using smooth functionals of the MLE. In our first example we consider the current status model described in Section 2 and estimate F_0 by the SMLE. In the second example we consider estimating a finite dimensional regression parameter for the current status model, where in addition to observing the vector (T, Δ) , also a covariate vector is observed.

4.1. The Smoothed Maximum Likelihood Estimator (SMLE)

We estimate F_0 by the SMLE \tilde{F}_{nh} obtained by first estimating the MLE F_n and then smoothing this using a smoothing kernel, i.e.,

$$\tilde{F}_{nh}(t) = \int IK((t-x)/h) dF_n(x), \quad (4.1)$$

where IK is an integrated kernel,

$$IK(u) = \int_{-\infty}^u K(x) dx,$$

and where h is a chosen bandwidth. Here dF_n represents the jumps of the discrete distribution function F_n and K is one of the usual symmetric twice differentiable kernels with compact support, used in density estimation. We use the triweight kernel

$$K(u) = \frac{35}{32} (1-u^2)^3 1_{[-1,1]}(u).$$

For a constant $c > 0$ and $h = cn^{-1/5}$, the SMLE [has been proved](#) to converge at rate $n^{-2/5}$ with asymptotic limit distribution,

$$n^{2/5} \left\{ \tilde{F}_{nh}(t) - F_0(t) \right\} \xrightarrow{\mathcal{D}} N(\beta, \sigma^2),$$

where

$$\beta = \frac{c^2 f_0'(t)}{2} \int u^2 K(u) du \quad \text{and} \quad \sigma^2 = \frac{F_0(t)\{1-F_0(t)\}}{cg(t)} \int K(u)^2 du. \quad (4.2)$$

(see [\[14\]](#)). The SMLE is often used in the smooth bootstrap procedures described in [Section 1](#) (see also the numerical example below). Let $\tilde{F}_{nh}^*(t)$ be the bootstrapped SMLE based on replacing F_n in [\(4.1\)](#) by the bootstrapped MLE \hat{F}_n , then we have the following result,

$$n^{2/5} \left\{ \tilde{F}_{nh}^*(t) - \tilde{F}_{nh}(t) \right\} \xrightarrow{\mathcal{D}} N(0, \sigma^2), \quad (4.3)$$

given the data $(T_1, \Delta_1), \dots, (T_n, \Delta_n)$, [in probability](#). [Omdat hier al \$n \rightarrow \infty\$, kun je volgens mij niet zeggen “with probability tending to one”](#). Note that, in contrast to the smooth bootstrap method described in [\[12\]](#), we do not need to estimate the convolution SMLE (see [\(4.6\)](#) below).

To prove the asymptotic normality result for the nonparametric bootstrap, given in [\(4.3\)](#), we prove (in [Section 6](#)) the following Lemma:

Lemma 4.1. *Assume that the conditions of [Lemma 3.1](#) are satisfied and that g has a bounded derivative g' on $[0, R]$. Let t be an interior point of $[0, R]$ such that f_0 has a continuous derivative f_0' at t . If $h \sim cn^{-1/5}$ then,*

$$\tilde{F}_{nh}^*(t) = \tilde{F}_{nh}^{(toy)*}(t) + o_{P_M}(n^{-2/5}),$$

in probability, where

$$\tilde{F}_{nh}^{(toy)*}(t) = \int IK((t-u)/h) dF_0(u) + \int \frac{K((t-u)/h) \{\delta - F_0(u)\}}{hg(u)} d\hat{\mathbb{P}}_n(u, \delta). \quad (4.4)$$

Since

$$\tilde{F}_{nh}(t) = \tilde{F}_{nh}^{(toy)}(t) + o_p(n^{-2/5}),$$

where $\tilde{F}_{nh}^{(toy)}(t)$ is defined by (4.4) with $\hat{\mathbb{P}}_n$ replaced by \mathbb{P}_n (see (11.53) on p. 335 of [15]), we have by Lemma 4.1 that,

$$n^{2/5} \left\{ \tilde{F}_{nh}^*(t) - \tilde{F}_{nh}(t) \right\} = n^{2/5} \int \frac{K((t-u)/h) \{\delta - F_0(u)\}}{hg(u)} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) + o_{P_M}(1),$$

in probability, which converges, conditional on the data $(T_1, \Delta_1), \dots, (T_n, \Delta_n)$ to the same asymptotic limit as

$$n^{2/5} \int \frac{K((t-u)/h) \{\delta - F_0(u)\}}{hg(u)} d(\mathbb{P}_n - P)(u, \delta),$$

in probability (see e.g. [17] for more details about the use of the bootstrap for kernel estimators). Finally, applying the central limit theorem on the expression above proves the asymptotic normality result for the bootstrapped SMLE given in (4.3).

Remark 4.1. In practice, one should use a boundary correction to ensure consistent estimation of F_0 near the boundaries of the support $[0, R]$. In our experiments we used the method of [25], see also p. 328 in [15]. It is straightforward to show that the nonparametric bootstrap method remains valid under this boundary correction. Moreover, one should also take into account the bias defined in (4.2) when constructing confidence intervals around the SMLE. Techniques to correct for the bias are discussed in [12] and will not be further elucidated in this paper.

To illustrate the performance of the nonparametric bootstrap procedure for constructing pointwise confidence intervals of the distribution function, we consider a simulation study based on $N = 5,000$ simulation runs from a model where both X and T have a Uniform(0,2) distribution. The $1 - \alpha$ bootstrap interval is given by:

$$\left[\tilde{F}_{nh}(t) - Q_{1-\alpha/2}^* \sqrt{S_{nh}(t)}, \tilde{F}_{nh}(t) + Q_{\alpha/2}^* \sqrt{S_{nh}(t)} \right], \quad (4.5)$$

where Q_α^* is the α th quantile of B values of $W_{nh}^*(t)$ defined by,

$$W_{nh}^*(t) = \left\{ \tilde{F}_{nh}^*(t) - \tilde{F}_{nh}(t) \right\} / \sqrt{S_{nh}^*(t)},$$

where $S_{nh}(t)$ resp. $S_{nh}^*(t)$ are estimates of the variance σ^2 defined in (4.2) (apart from the factor $cg(t)$ which drops out in the Studentized bootstrap procedure) given by:

$$S_{nh}(t) = \frac{1}{n^2} \sum_{i=1}^n K_h(t - T_i)^2 (\Delta_i - F_n(T_i))^2,$$

$$S_{nh}^*(t) = \frac{1}{n^2} \sum_{i=1}^n M_{ni} K_h(t - T_i)^2 (\Delta_i - \hat{F}_n(T_i))^2.$$

In Figure 2(a) we compare the proportion of times that $F_0(t_i)$ is not in the 95% bootstrap confidence intervals for $t_i = 0.02, 0.04, \dots, 2$ with the corresponding proportions obtained with the smooth bootstrap procedure proposed in [12]. For samples of size $n = 1,000$, $B = 1,000$ bootstrap samples were generated for both methods and the triweight kernel is used for calculation of the SMLE with $h = 2n^{-1/5}$. For the smooth bootstrap procedure, first a bootstrap sample $(T_1, \Delta_1^*), \dots, (T_n, \Delta_n^*)$ is obtained by keeping the T_i in the original sample fixed and by resampling the Δ_i^* from a Bernoulli distribution with probability $\tilde{F}_{nh}(T_i)$, then the bootstrap MLE \hat{F}_n and SMLE \tilde{F}_{nh}^* are estimated based on the $(T_i, \Delta_i^*), i = 1, \dots, n$. The smooth bootstrap $1 - \alpha$ intervals are then constructed via (4.5), except that the SMLE $\tilde{F}_{nh}(t)$ in the definition of $W_{nh}^*(t)$ is replaced by the convolution SMLE given by,

$$\int IK_h(t - u) d\tilde{F}_{nh}(u), \quad (4.6)$$

and that the variance estimate in the bootstrap sample is given by:

$$\frac{1}{n^2} \sum_{i=1}^n K_h(t - T_i)^2 (\Delta_i^* - \hat{F}_n(T_i))^2.$$

The convolution SMLE corresponds to the extra level of smoothing introduced by the smooth bootstrap procedure and is hence not required for the nonparametric bootstrap.

Figure 2(b) shows the average length of both bootstrap intervals in comparison with the average length of the Banerjee-Wellner ([3]) and Sen-Xu ([26]) CIs. The latter intervals are constructed around the MLE F_n instead of the SMLE \tilde{F}_{nh} . The length of the MLE-based intervals is larger than the length of the SMLE-based intervals due to the fact that the MLE converges at the slower rate $n^{1/3}$. The performance of the SMLE-based CIs is comparable. The bootstrap intervals based on the classical bootstrap procedure avoid however calculation of the convolution SMLE defined in (4.6).

We also applied the bootstrap procedures to the Rubella data set described by [19]. The data set contains 230 observations on the prevalence of rubella in Austrian males. For the smooth bootstrap, confidence intervals were calculated in [12] based on the local bandwidth $h(t_i) = (0.25M + t_i)n^{-1/5}$ if $t_i \leq 20$ and $h(t_i) = h(20) + 2(t_i - 20)n^{-1/5}$ else at time points $t_1 = M/100, t_2 =$

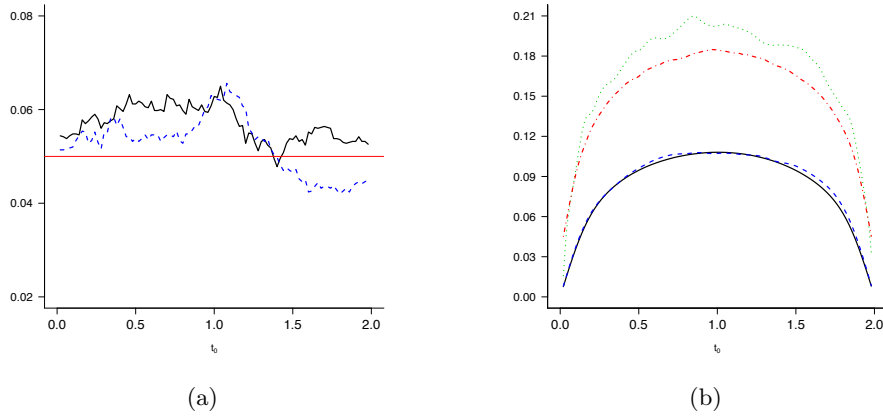


Fig 2: Uniform samples: (a) Proportion of times that $F_0(t_i)$, $t_i = 0.02, 0.04, \dots$ is not in the 95% CIs and (b) average length of the CIs for the classical bootstrap (black, solid) and the smooth bootstrap (blue, dashed) procedure in constructing CIs around the SMLE. In (b) the average length for the Banerjee-Wellner (red, dashed-dotted) and Sen-Xu (green, dotted) CIs around the MLE is also shown. $n = 1,000$, $N = 5,000$, $B = 1,000$ and $h = 2n^{-1/5}$.

$2M/100, \dots, M$ where $M = 80.1178$ is the largest observed age. The local bandwidth takes into account the bias of the SMLE given in (4.2). Figure 3 shows the confidence intervals obtained with the nonparametric bootstrap and illustrates the applicability of our method in a real data example.

4.2. The current status linear regression model

In the current status linear regression model we are interested in the estimation of the regression parameter β_0 based on observations $(T_1, X_1, \Delta_1 = 1_{\{Y_1 \leq T_1\}}), \dots, (T_n, X_n, \Delta_n = 1_{\{Y_n \leq T_n\}})$ from (T, X, Δ) where we assume that

$$Y_i = \beta'_0 X_i + \varepsilon_i \quad i = 1, 2, \dots$$

with i.i.d. random error terms ε_i , independent of (T_i, X_i) with unknown distribution function F_0 .

In [13] a simple score estimator β_n was introduced depending on the MLE $F_{n,\beta}$, defined as,

$$F_{n,\beta} \stackrel{def}{=} \arg \max_{F_\beta \in \mathcal{F}} \sum_{i=1}^n [\Delta_i \log F_\beta(T_i - \beta' X_i) + (1 - \Delta_i) \log \{1 - F_\beta(T_i - \beta' X_i)\}], \tag{4.7}$$

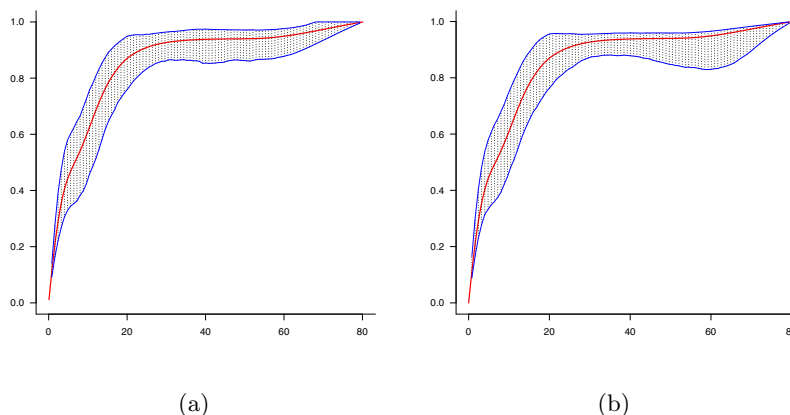


Fig 3: Rubella data: SMLE (red, solid) and (a) CI defined in (4.5) and (b) smooth CI proposed by [12] based on $n = 230$ observations using $B = 1,000$ bootstrap samples and adaptive bandwidth $h(t) = (0.25M + t)n^{-1/5}$ if $t \leq 20$ and $h(t) = h(20) + 2(t - 20)n^{-1/5}$ else.

where $\mathcal{F} = \{F : \mathbb{R} \mapsto [0, 1] : F \text{ is a distribution function}\}$. The estimator β_n for β_0 is next defined as the root of

$$\sum_{F_{n,\beta}(T_i - \beta'X_i) \in [\epsilon, 1-\epsilon]} X_i \{F_{n,\beta}(T_i - \beta'X_i) - \Delta_i\} = 0, \quad (4.8)$$

for some fixed truncation parameter $\epsilon \in (0, 1/2)$. It is proved in [13] that $\sqrt{n}\{\beta_n - \beta_0\}$ is asymptotically normal with mean zero and variance $V^{-1}WV^{-1}$ where

$$V = E_\epsilon \left[f_0(T - \beta'_0 X) \{X - E(X|T - \beta'_0 X)\} \{X - E(X|T - \beta'_0 X)\}' \right],$$

$$W = E_\epsilon \left[F_0(T - \beta'_0 X) \{1 - F_0(T - \beta'_0 X)\} \{X - E(X|T - \beta'_0 X)\} \times \{X - E(X|T - \beta'_0 X)\}' \right],$$

where $E_\epsilon(w(T, X, \Delta)) = \int_{F_0(t - \beta'_0 x) \in [\epsilon, 1-\epsilon]} w(t, x, \delta) dP(t, x, \delta)$ is the truncated expectation of $w(T, X, \Delta)$ for some deterministic function w and where P denotes the probability measure of (T, X, Δ) .

A bootstrap version $\hat{\beta}_n$ based on a bootstrap sample from \mathbb{P}_n is then defined as the root of

$$\sum_{\hat{F}_{n,\beta}(T_i - \beta'X_i) \in [\epsilon, 1-\epsilon]} M_{ni} X_i \{\hat{F}_{n,\beta}(T_i - \beta'X_i) - \Delta_i\} = 0, \quad (4.9)$$

where $\hat{F}_{n,\beta}$ is the MLE in the bootstrap sample. A straightforward extension of the results given in Section 3 shows that, as n tends to infinity,

$$E_{M|Z} \left| n^{-1/3} \left\{ \hat{F}_{n,\beta}(t - \beta'x) - F_\beta(t - \beta'x) \right\} \right|^p,$$

stays bounded in probability for all $(t, x) \in \{(t, x) : F_\beta(t - \beta'x) \in [\epsilon, 1 - \epsilon]\}$ and for all β in a neighborhood of β_0 where F_β is defined by,

$$F_\beta(u) = P \{ \Delta_i = 1 \mid T_i - \beta'X_i = u \} = \int F_0(u + (\beta - \beta_0)'x) f_{X|T-\beta'X}(x|u) dx. \tag{4.10}$$

The validity of the bootstrap method follows from the fact that, in probability, we have conditionally on the data $(T_1, X_1, \Delta_1), \dots, (T_n, X_n, \Delta_n)$ that,

$$\begin{aligned} \sqrt{n}V(\hat{\beta}_n - \beta_n) &= \sqrt{n} \int_{F_0(t-\beta'_0x) \in [\epsilon, 1-\epsilon]} \{x - E(X|T - \beta'_0X = t - \beta'_0x)\} \\ &\quad \cdot \{F_0(t - \beta'_0x) - \delta\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(t, x, \delta) \\ &\quad + o_{P_M}(1 + \sqrt{n}(\hat{\beta}_n - \beta_n)), \end{aligned} \tag{4.11}$$

where the dominant term in the right-hand side of the display above is normally distributed with mean zero and variance W conditional on $(T_1, X_1, \Delta_1), \dots, (T_n, X_n, \Delta_n)$.

Remark 4.2. The nonparametric bootstrap is also valid for the second estimator of β_0 proposed in [13] based on a different score function involving the MLE $\hat{F}_{n,\beta}$ and the derivative of the SMLE $\hat{F}_{nh,\beta}$ (constructed by the procedure described in previous Section 4.1).

To provide more insight in the finite sample behavior of the classical bootstrap estimators we show in Tables 1 and 2 the results of two simulation studies for a one-dimensional regression model $Y = \beta_0X + \varepsilon$. In the first simulation setting we take $\beta_0 = 0.5$ and consider Uniform(0,2) distributions for the variables T and X ; for the distribution of the random error ε we take $f_0(e) = 384(e - 3/8)(5/8 - e)1_{[3/8, 5/8]}(e)$. A picture of the density and distribution function of the random error in model 1 is shown in Figure 4. The first model is also analyzed in [13]. In the second simulation model T, X and ε are independently sampled from a standard normal distribution and $\beta_0 = 1$. A similar model was considered in [1].

With these simulations we want to point out that it is not necessary to use smoothing techniques for doing inferences in the current status linear regression model. We compare the simple score estimator (SSE) described above with Han's maximum rank correlation estimator ([18], MRCE) and with the efficient score estimator (ESE) proposed in [13]. The asymptotic behavior of the MRCE for the current status model, also obtained without any smoothing techniques, is established in [1] where the author also proposes consistent kernel-based estimates of the asymptotic variance of the MRCE. We use these variance estimates to construct estimates for V, W and the almost (determined by the truncation

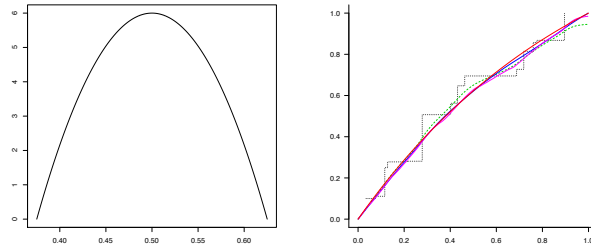


Fig 4: The density f_0 (left panel) and distribution function F_0 (right panel) of the random error ε in simulation model 1.

parameter ϵ) efficient variance of the SSE. For more details about the variance estimation we refer to [1].

A summary of $N = 1,000$ simulation runs from models 1 and 2 for different sample sizes n is given in Tables 1 and 2. For each estimator, the mean, n times the variance and n times the mean squared-error (MSE) is given in columns 3-5. The asymptotic variance of the estimators equals 0.193612 for the SSE, 0.158699 for the ESE and 0.192857 for the MRCE in model 1 using truncation parameter $\epsilon = 0.001$. The corresponding asymptotic variances in model 2 equal 5.046413, 4.994988 and 5.35448 respectively. The asymptotic variance of the SSE without truncation (i.e. $\epsilon = 0$) equals the asymptotic variance of the MRCE in model 1. The efficient variances are 0.151706 in model 1 and 4.994987 in model 2. Note that the differences between the limiting variances for the different estimation methods are tiny and that the effect of the truncation parameter ϵ on the asymptotic behavior of the score estimators is small. Tables 1 and 2 show that n times the variance tends to converge to the asymptotic variance for all estimators. The ESE performs worse for small sample sizes and the results suggest to use the SSE for point estimation of the regression parameter β_0 .

We constructed Wald-type confidence intervals, similar to the intervals proposed in [1], using the asymptotic normal limiting distribution of the estimators and compared the coverage proportion and average length of these intervals with bootstrap confidence intervals based on the nonparametric bootstrap described in this paper using $B = 1,000$ samples from the original data. For the MRCE, the validity of the classical bootstrap is proved in [27]. The Wald-type confidence intervals remain anti-conservative for the ESE in model 2.

We observed (result not shown) that, in both models, the bias in estimating the efficient variance of the ESE remains larger than the bias of the asymptotic variance estimates for the SSE and the MRCE. Tables 1 and 2 show that the coverage proportion of the classical bootstrap confidence intervals converges to the nominal 95%–level and the average length of the confidence intervals obtained by resampling from the original data is smaller than the corresponding length of the Wald-type confidence intervals. We also investigated the behavior

of Studentized bootstrap confidence intervals (results not shown) based on the variance estimate used in the construction of the Wald-type confidence intervals, but no improvement was observed for the behavior of the bootstrap intervals.

Our results do not indicate better performances corresponding to smoothing techniques and therefore suggest that smoothing should not be the primary concern in inferences for the current status linear regression model. Note that the Wald-type confidence intervals are constructed using smoothing kernel estimation for the variance estimate and that the only results obtained without any smoothing are the bootstrap confidence interval for the SSE and the MRCE. It is noteworthy that the SSE tends to perform better than the MRCE, which is not based on a nuisance parameter that is not estimable at \sqrt{n} -rate. Based on these results, we recommend the use of the SSE in combination with the non-parametric bootstrap procedure for doing inference in the current status linear regression model.

TABLE 1

Simulation model 1: mean, n times the variance and n times MSE. CP: coverage proportion of 95% confidence intervals (Wald-type intervals based on a kernel variance estimate and classical bootstrap intervals) that contain the true parameter value $\beta_0 = 0.5$, AL: Average length of the CI, for different samples sizes n based on $N = 1,000$ simulation runs and $B = 1,000$ bootstrap samples. $\epsilon = 0.001$. SSE = simple score estimator, MRCE = maximum rank correlation estimator and ESE = efficient score estimator.

Estimate	n	mean	$n \times \text{var}$	$n \times \text{MSE}$	Wald-type CI		Bootstrap CI	
					CP	AL	CP	AL
SSE	100	0.498943	0.310723	0.310968	0.978	0.265883	0.824	0.204163
	500	0.499717	0.220885	0.220925	0.982	0.097457	0.897	0.080317
	1000	0.500720	0.217415	0.217933	0.977	0.065837	0.924	0.055648
	5000	0.499993	0.195111	0.195112	0.977	0.027159	0.945	0.024423
MRCE	100	0.497996	0.308180	0.308582	0.979	0.268731	0.821	0.205522
	500	0.499761	0.251232	0.251260	0.978	0.098028	0.862	0.089143
	1000	0.500553	0.246388	0.246693	0.973	0.063990	0.911	0.053129
	5000	0.499876	0.208386	0.208462	0.965	0.027197	0.922	0.026987
ESE	100	0.500145	0.337755	0.337757	0.964	0.252687	0.824	0.223849
	500	0.499671	0.217428	0.217482	0.978	0.094390	0.896	0.080003
	1000	0.500742	0.207401	0.207953	0.973	0.063990	0.911	0.053129
	5000	0.500228	0.185614	0.185874	0.972	0.026396	0.904	0.022285

5. Discussion

In this paper we studied the behavior of the nonparametric bootstrap in current status models. Asymptotic results show that, given the data, the L_2 -distance between the bootstrap MLE \hat{F}_n and the underlying distribution function F_0 is of order $n^{-1/3}$. This result is noteworthy seen the fact that the nonparametric bootstrap is inconsistent for generating the distribution of the MLE. Despite this negative result, we show that it is still possible to use the MLE while doing inferences for certain functionals in the current status model. We illustrated the effectiveness of this result by constructing pointwise confidence intervals

TABLE 2

Simulation model 2: mean, n times the variance and n times MSE. CP: coverage proportion of 95% confidence intervals (Wald-type intervals based on a kernel variance estimate and classical bootstrap intervals) that contain the true parameter value $\beta_0 = 1$, AL: Average length of the CI, for different samples sizes n based on $N = 1,000$ simulation runs and $B = 1,000$ bootstrap samples. $\epsilon = 0.001$. SSE = simple score estimator, MRCE = maximum rank correlation estimator and ESE = efficient score estimator.

Estimate	n	mean	$n \times \text{var}$	$n \times \text{MSE}$	Wald-type CI		Bootstrap CI	
					CP	AL	CP	AL
SSE	100	0.935732	4.525330	4.938096	0.922	1.000283	0.855	0.79952
	500	0.966217	4.676249	5.246881	0.926	0.399728	0.902	0.364210
	1000	0.977799	5.032432	5.525339	0.933	0.279928	0.914	0.262449
	5000	0.989466	4.580756	5.135616	0.945	0.124375	0.948	0.121388
MRCE	100	1.038510	8.500588	8.648890	0.925	1.125225	0.889	1.364034
	500	1.006050	6.443404	6.461690	0.932	0.429007	0.912	0.473787
	1000	1.002680	6.294143	6.301326	0.939	0.296537	0.903	0.320908
	5000	0.998502	5.160694	5.171915	0.962	0.129512	0.954	0.136487
ESE	100	0.974199	5.722576	5.789144	0.768	0.604649	0.827	0.910229
	500	0.998806	5.984291	5.985003	0.823	0.290297	0.902	0.430819
	1000	1.005545	6.032743	6.063495	0.841	0.214280	0.928	0.302124
	5000	1.002462	5.244373	5.274692	0.892	0.104281	0.951	0.131427

around the SMLE and proved the validity of interval estimation in the current status linear regression model. The result is however applicable to several other estimators depending on a cube-root n convergence class. Because of its connection with the MLE, applications of the nonparametric bootstrap involving the Grenander estimator, such as the smoothed Grenander estimator used in [5] or the goodness-of-fit tests described in [6], are worthy of study in further research.

Probably similar results will follow for the more challenging interval censoring, type II models where the development of the local limit theory for the MLE has not yet been settled. It is reasonable to believe that the nonparametric bootstrap also allows for inferences with the maximum smoothed likelihood estimator studied in [10].

The programs to produce the results presented in this paper can be found in [11].

6. Appendix

6.1. Proof of Lemma 3.1

Before proving Lemma 3.1 we provide two technical lemmas.

Lemma 6.1. *Let $\alpha > 0$. There exist constants $K_1, K_2 > 0$ such that, for each*

$j \geq 1, j \in \mathbb{N}$,

$$P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}) : \left| \int_{u \in (U(a), U(a)+y]} \{\delta - F_0(u)\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \right| \geq \alpha(j-1)^2 n^{-2/3} \right\} \leq K_1 \exp \left\{ -K_2(j-1)^{3/2} \right\}, \quad (6.1)$$

in probability.

Likewise, there exist constants $K_1, K_2 > 0$ such that, for each $j \geq 1, j \in \mathbb{N}$,

$$P_{M|Z} \left\{ \exists y \in [-jn^{-1/3}, -(j-1)n^{-1/3}) : \left| \int_{u \in (U(a)+y, U(a)]} \{\delta - F_0(u)\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \right| \geq \alpha(j-1)^2 n^{-2/3} \right\} \leq K_1 \exp \left\{ -K_2(j-1)^{3/2} \right\}, \quad (6.2)$$

in probability.

Proof. We only prove (6.1), since the proof of (6.2) is similar. Let \mathcal{F}_t be the (Vapnik-Cervonenkis) class of functions

$$\mathcal{F}_t = \{(\delta - F_0(v))1_{(U(a), U(a)+u]}(v) : u \in [0, t], \delta \in \{0, 1\}\},$$

with envelope

$$F_t(v, \delta) = 1_{(U(a), U(a)+t]}(v), \quad v \in [0, t].$$

To prove (6.1), we use that an exponential tail bound can be derived from a bounded Orlicz norm $\|\cdot\|_{P, \psi}$, i.e., when taking $\psi_1(x) = \exp(x) - 1$, for $x \geq 0$, we get, for $x > 0$ the inequality

$$P(|X| > x) \leq 2 \exp \{-x/\|X\|_{P, \psi_1}\}, \quad (6.3)$$

where

$$\|X\|_{P, \psi_1} = \inf \left\{ C > 0 : E\psi_1 \left(\frac{|X|}{C} \right) \leq 1 \right\}.$$

Using the second statement of Theorem 2.14.5 in [28], with $p = 1$, we get, the following inequality:

$$\begin{aligned} & \left\| \left\| \sqrt{n} \left(\hat{\mathbb{P}}_n - \mathbb{P}_n \right) \right\|_{\mathcal{F}_t}^* \right\|_{\mathbb{P}_n, \psi_1} \\ & \lesssim \left\| \left\| \sqrt{n} \left(\hat{\mathbb{P}}_n - \mathbb{P}_n \right) \right\|_{\mathcal{F}_t}^* \right\|_{\mathbb{P}_n, 1} + n^{-1/2} \{1 + \log n\} \|F_t\|_{\mathbb{P}_n, \psi_1}, \end{aligned} \quad (6.4)$$

where $\|\cdot\|_{\mathcal{F}_t}^*$ denotes the so-called measurable majorant of $\|\cdot\|_{\mathcal{F}_t}$ (see [28]). (Note that we use temporarily the ”*” notation which is used for bootstrap variables in the rest of the paper.)

Furthermore, we have by the rightmost inequality of Theorem 2.14.1 of [28] that

$$\left\| \left\| \sqrt{n} \left(\hat{\mathbb{P}}_n - \mathbb{P}_n \right) \right\|_{\mathcal{F}_t}^* \right\|_{\mathbb{P}_n,1} \lesssim J(1, \mathcal{F}_t) \|F_t\|_{\mathbb{P}_n,2},$$

where $J(\delta, \mathcal{F}_t)$ is defined by

$$J(\delta, \mathcal{F}_t) = \sup_Q \int_0^\delta \sqrt{1 + \log N(\varepsilon \|F\|_{Q,2}, \mathcal{F}_t, L_2(Q))} d\varepsilon,$$

and where the supremum is over all discrete probability measure Q with $\|F_t\|_{Q,2} > 0$. Since $\mathcal{F}_t \subset \mathcal{F}_{R-U(a)}$ for all $t \in [0, R - U(a)]$, and since $\mathcal{F}_{R-U(a)}$ is a Vapnik-Cervonenkis class, $J(\delta, \mathcal{F}_t)$ is bounded by a fixed constant for all $t \in [0, R - U(a)]$, and we get:

$$\left\| \left\| \sqrt{n} \left(\hat{\mathbb{P}}_n - \mathbb{P}_n \right) \right\|_{\mathcal{F}_t}^* \right\|_{\mathbb{P}_n,1} \lesssim \|F_t\|_{\mathbb{P}_n,2},$$

uniformly for all $t \in [0, R - U(a)]$. Note that

$$\|F_t\|_{\mathbb{P}_n,2}^2 = \int_{u \in U(a), U(a)+t] d\mathbb{P}_n(u, \delta) = \int_{u \in U(a), U(a)+t] d\mathbb{G}_n(u), \quad (6.5)$$

$t \in [U(a), R - U(a)]$. We next evaluate the second term on the right-hand side of (6.4). We have:

$$\int \psi_1 \left(\frac{F_t(u, \delta)}{c} \right) d\mathbb{P}_n(u, \delta) = \left\{ e^{1/c} - 1 \right\} \int 1_{(U(a), U(a)+t]}(u) d\mathbb{G}_n(u),$$

and

$$\begin{aligned} \left\{ e^{1/c} - 1 \right\} \int 1_{(U(a), U(a)+t]}(u) d\mathbb{G}_n(u) &\leq 1 \\ \iff c &\geq \frac{1}{\log \left\{ 1 + 1 / \int_{u \in U(a), U(a)+t]} d\mathbb{G}_n(u) \right\}}. \end{aligned}$$

Thus (6.4) becomes, using (6.5),

$$\begin{aligned} &\left\| \left\| \sqrt{n} \left(\hat{\mathbb{P}}_n - \mathbb{P}_n \right) \right\|_{\mathcal{F}_t}^* \right\|_{\mathbb{P}_n, \psi_1} && (6.6) \\ &\leq c_1 \left\{ \int_{u \in U(a), U(a)+t]} d\mathbb{G}_n(u) \right\}^{1/2} + \frac{1 + \log n}{n^{1/2} \log \left\{ 1 + 1 / \int_{u \in U(a), U(a)+t]} d\mathbb{G}_n(u) \right\}}, && (6.7) \end{aligned}$$

for a constant $c_1 > 0$. If $t \geq Kn^{-1/3}$ we get for the second term in probability,

$$\frac{1 + \log n}{n^{1/2} \log \left\{ 1 + 1 / \int_{u \in U(a), U(a)+t} d\mathbb{G}_n(u) \right\}} \ll c_1 \left\{ \int_{u \in U(a), U(a)+t} d\mathbb{G}_n(u) \right\}^{1/2}.$$

We have:

$$\begin{aligned} & \int_{u \in [U(a), U(a)+t]} d\mathbb{G}_n(u) \\ &= \int_{u \in [U(a), U(a)+t]} dG(u) + \int_{u \in [U(a), U(a)+t]} d(\mathbb{G}_n - G)(u) \\ &= \int_{u \in [U(a), U(a)+t]} dG(u) + O_p(n^{-1/2}) = O(t) + O_p(n^{-1/2}) \\ &= O(t) + O_{P_M}(n^{-1/2}), \end{aligned}$$

in probability (since a term defined only on the probability space $(\mathcal{X}, \mathcal{A}, P)$ of order $O_p(1)$ is also of order $O_{P_M}(1)$ in probability). So we obtain, for $j \geq K$ in probability, conditioning on $(T_1, \Delta_1), (T_2, \Delta_2), \dots$ using the inequality on Orlicz norms on p. 96 or 239 of [28]:

$$\begin{aligned} & P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}] : \right. \\ & \quad \left. \left| \int_{u \in (U(a), U(a)+y]} \{\delta - F_0(u)\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \right| \geq \alpha(j-1)^2 n^{-2/3} \right\} \\ &= P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}] : \right. \\ & \quad \left. \sqrt{n} \left| \int_{u \in (U(a), U(a)+y]} \{\delta - F_0(u)\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \right| \geq \alpha(j-1)^2 n^{-1/6} \right\} \\ &\leq 2 \exp \left\{ -m(j-1)^2 n^{-1/6} / \left\| \left\| \sqrt{n} (\hat{\mathbb{P}}_n - \mathbb{P}_n) \right\|_{\mathcal{F}_{jn^{-1/3}}}^* \right\|_{\mathbb{P}_n, \psi_1} \right\} \\ &\leq 2 \exp \left\{ -c_2 m(j-1)^{3/2} \right\}, \end{aligned}$$

for some $c_2 > 0$. This proves the statement. \square

Lemma 6.2. For each $\varepsilon > 0$ and $x \in [0, R - U(a)]$,

$$\left| \int_{u \in (U(a), U(a)+x]} \{\delta - F_0(u)\} d(\mathbb{P}_n - P)(u, \delta) \right| \leq \varepsilon x^2 + O_p(n^{-2/3}).$$

Proof. As in the proof of Lemma 6.1, we consider the Vapnik-Cervonenkis collection of functions:

$$\mathcal{F}_t = \{(\delta - F_0(v))1_{(U(a), U(a)+u]}(v) : u \in [0, t], \delta \in \{0, 1\}\},$$

with envelope

$$F_t(v, \delta) = 1_{(U(a), U(a)+t]}(v), \quad v \in [0, t].$$

We have, using Theorem 2.14.1 of [28]:

$$E_X \left\{ \sup_{f \in \mathcal{F}_t} |\mathbb{P}_n - P|(f) \right\}^2 \leq Kn^{-1} \|F_t\|_{P,2}^2, \quad (6.8)$$

for some $K > 0$. Since,

$$\|F_t\|_{P,2}^2 = \int_{u \in U(a), U(a)+t]} dP(u, \delta) = \int_{u \in U(a), U(a)+t]} dG(u) = O(t),$$

for $t \in [U(a), R - U(a)]$, we get, by Markov's inequality,

$$\begin{aligned} & P \left\{ n^{2/3} \left| \int_{u \in (U(a), U(a)+jn^{-1/3}]} \{\delta - F_0(u)\} d(\mathbb{P}_n - P)(u, \delta) \right| > A + \varepsilon(j-1)^2 \right\} \\ & \leq Kj / \{A + \varepsilon(j-1)^2\}^2. \end{aligned}$$

The result now easily follows, see, e.g., [20]. p. 201. \square

As a consequence of Lemma 6.1 and Lemma 6.2 we get the following result.

Lemma 6.3. *Let \hat{V}_n and $\hat{\hat{V}}_n$ be defined by*

$$\hat{V}_n(t) = \int_{u \in [0, t]} \delta d\hat{\mathbb{P}}_n(u, \delta), \quad \hat{\hat{V}}_n(t) = \int_{u \in [0, t]} F_0(u) d\hat{\mathbb{G}}_n(u), \quad t \in [0, R]. \quad (6.9)$$

where the process $\hat{\mathbb{G}}_n$ is defined in (3.3), and let $\hat{D}_n = \hat{V}_n - \hat{\hat{V}}_n$. Then there exist constants $K_1, K_2 > 0$ such that, for each $j \geq 1, j \in \mathbb{N}$,

$$\begin{aligned} & P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}] : \hat{D}_n(U(a)+y) - \hat{D}_n(U(a)) \right. \\ & \qquad \qquad \qquad \leq - \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d\hat{\mathbb{G}}_n(u) \left. \right\} \\ & \leq K_1 \exp \left\{ -K_2(j-1)^{3/2} \right\}, \quad (6.10) \end{aligned}$$

in probability. Likewise, there exist constants $K_1, K_2 > 0$ such that, for each $j \geq 1, j \in \mathbb{N}$,

$$\begin{aligned} P_{M|Z} \left\{ \exists y \in [-jn^{-1/3}, -(j-1)n^{-1/3}] : \hat{D}_n(U(a) + y) - \hat{D}_n(U(a)) \right. \\ \left. \leq - \int_{U(a)+y}^{U(a)} \{F_0(u) - F_0(U(a))\} d\hat{\mathbb{G}}_n(u) \right\} \\ \leq K_1 \exp \left\{ -K_2(j-1)^{3/2} \right\}, \end{aligned} \quad (6.11)$$

in probability.

Proof. We again only prove (6.1), since the proof of (6.2) is similar. First note:

$$\begin{aligned} P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}] : \hat{D}_n(U(a) + y) - \hat{D}_n(U(a)) \right. \\ \left. \leq - \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d\hat{\mathbb{G}}_n(u) \right\} \\ \leq P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}] : \left| \hat{D}_n(U(a) + y) - \hat{D}_n(U(a)) \right| \right. \\ \left. \geq \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d\hat{\mathbb{G}}_n(u) \right\}. \end{aligned}$$

Furthermore:

$$\begin{aligned} \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d\hat{\mathbb{G}}_n(u) \\ = \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d\mathbb{G}_n(u) \\ + \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d(\hat{\mathbb{G}}_n - \mathbb{G}_n)(u) \\ = \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} dG(u) \\ + \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d(\mathbb{G}_n - G)(u) \\ + \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d(\hat{\mathbb{G}}_n - \mathbb{G}_n)(u), \end{aligned} \quad (6.12)$$

and for the dominant term on the right-hand side we get:

$$\begin{aligned} \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} dG(u) &\geq m_0 \int_{U(a)}^{U(a)+y} \{u - U(a)\} dG(u) \\ &\geq m_0 m_1 \int_{U(a)}^{U(a)+y} \{u - U(a)\} du = \frac{1}{2} m_0 m_1 \{y - U(a)\}^2, \end{aligned}$$

where $m_0 = \inf_{u \in [U(a), R]} f_0(u)$ and $m_1 = \inf_{u \in [U(a), R]} g(u)$. We therefore consider the probability:

$$\begin{aligned} P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}] : \left| \hat{D}_n(U(a)+y) - \hat{D}_n(U(a)) \right| \right. \\ \left. \geq m(j-1)^2 n^{-2/3} \right\}. \end{aligned} \quad (6.13)$$

where

$$m = \frac{1}{2} \min \left\{ \inf_{u \in [t_0, R]} f_0(u), \inf_{u \in [t_0, R]} g(u) \right\}.$$

We also have:

$$\begin{aligned} \hat{D}_n(U(a)+y) - \hat{D}_n(U(a)) &= \int_{u \in (U(a), U(a)+y]} \{\delta - F_0(u)\} d\hat{\mathbb{P}}_n(u, \delta) \\ &= \int_{u \in (U(a), U(a)+y]} \{\delta - F_0(u)\} d(\hat{\mathbb{P}}_n - P)(u, \delta) \\ &= \int_{u \in (U(a), U(a)+y]} \{\delta - F_0(u)\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \\ &\quad + \int_{u \in (U(a), U(a)+y]} \{\delta - F_0(u)\} d(\mathbb{P}_n - P)(u, \delta). \end{aligned}$$

By Lemma 6.2, we may assume that for $x \in [0, R - U(a)]$,

$$\left| \int_{u \in (U(a), U(a)+x]} \{\delta - F_0(u)\} d(\mathbb{P}_n - P)(u, \delta) \right| \leq \varepsilon x^2 + Kn^{-2/3}, \quad (6.14)$$

for some $K > 0$ and $0 < \varepsilon < m/2$. Considering sequences $X = (T_1, \Delta_1), (T_2, \Delta_2) \dots$, satisfying (6.14), we get:

$$\begin{aligned} P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}] : \left| \hat{D}_n(U(a)+y) - \hat{D}_n(U(a)) \right| \right. \\ \left. \geq m(j-1)^2 n^{-2/3} \right\} \\ \leq P_{M|Z} \left\{ \exists y \in [(j-1)n^{-1/3}, jn^{-1/3}] : \right. \\ \left. \left| \int_{u \in (U(a), U(a)+y]} \{\delta - F_0(u)\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \right| \geq \frac{1}{2} m(j-1)^2 n^{-2/3} \right\} \\ \leq K_1 \exp \left\{ -K_2(j-1)^{3/2} \right\}, \end{aligned}$$

with probability tending to one, using Lemma 6.1. \square

We now prove Lemma 3.1.

Proof of Lemma 3.1. Suppose that $n^{1/3}|\hat{U}_n(a) - U(a)| > x$ for some $x > 0$, then there exists a y such that, $n^{1/3}|y - U(a)| > x$ and $\hat{V}_n(y) - a\hat{G}_n(y) \leq \hat{V}_n(U(a)) - a\hat{G}_n(U(a))$. Hence,

$$\begin{aligned} & P_{M|Z} \left\{ n^{1/3} \left| \hat{U}_n(a) - U(a) \right| \geq x \right\} \\ & \leq P_{M|Z} \left(\inf_{y-U(a) \geq n^{-1/3}x} \hat{D}_n(y) - \hat{D}_n(U(a)) \right. \\ & \qquad \qquad \qquad \left. \leq - \int_{U(a)}^y \{F_0(u) - F_0(U(a))\} d\hat{G}_n(u) \right) \\ & \leq \sum_{j=i}^{\infty} P_{M|Z} \left(\exists y \in [(j-1)n^{-1/3}, jn^{-1/3}) : \hat{D}_n(U(a) + y) - \hat{D}_n(U(a)) \right. \\ & \qquad \qquad \qquad \left. \leq - \int_{U(a)}^{U(a)+y} \{F_0(u) - F_0(U(a))\} d\hat{G}_n(u) \right), \end{aligned}$$

where $x \in [(i-1)n^{-1/3}, in^{-1/3}]$. By Lemma 6.3, this is bounded above by,

$$\begin{aligned} & \sum_{j=i}^{\infty} K_1 \exp \left\{ K_2(j-1)^{3/2} \right\} \\ & = K_1 \exp \left\{ -K_2(i-1)^{3/2} \right\} \sum_{j=i}^{\infty} \exp \left\{ -K_2[(j-1)^{3/2} - (i-1)^{3/2}] \right\} \\ & \leq K'_1 \exp \left\{ K'_2(i-1)^{3/2} \right\}, \end{aligned}$$

for constants $K_1, K'_1, K_2, K'_2 > 0$. \square

6.2. Proof of Lemma 4.1

We introduce notations K_h and IK_h to denote the scaled versions of K and IK respectively:

$$K_h(u) = h^{-1}K(u/h) \quad \text{and} \quad IK_h(u) = IK(u/h).$$

Proof. Define the function

$$\psi_{t,h}(u) = \frac{K_h(t-u)}{g(u)}.$$

Denote the points of jump of the MLE \hat{F}_n by $\hat{\tau}_1, \dots, \hat{\tau}_m$ and define the piecewise constant function $\bar{\psi}_{t,h}$ with only jumps at $\hat{\tau}_1, \dots, \hat{\tau}_m$ by

$$\bar{\psi}_{t,h}(u) = \begin{cases} \psi_{t,h}(\hat{\tau}_i), & \text{if } F_0(u) > \hat{F}_n(\hat{\tau}_i), u \in [\hat{\tau}_i, \hat{\tau}_{i+1}), \\ \psi_{t,h}(s), & \text{if } F_0(u) = \hat{F}_n(s), \text{ for some } s \in [\hat{\tau}_i, \hat{\tau}_{i+1}), \\ \psi_{t,h}(\hat{\tau}_{i+1}), & \text{if } \hat{F}_0(u) < \hat{F}_n(\tau_i), u \in [\hat{\tau}_i, \hat{\tau}_{i+1}). \end{cases}$$

By the convex minorant interpretation of \hat{F}_n , we have

$$\int \bar{\psi}_{t,h}(u)(\delta - \hat{F}_n(u))d\hat{\mathbb{P}}_n(u, \delta) = 0,$$

(see the discussion of the SMLE in [15], p. 332).

We can write

$$\begin{aligned} \tilde{F}_{nh}^*(t) &= \int IK_h(t-u) d\hat{F}_n(u) \\ &= \int IK_h(t-u) d(\hat{F}_n - F_0)(u) + \int IK_h(t-u) dF_0(u) \\ &= \int \psi_{t,h}(u) \left\{ \hat{F}_n(u) - F_0(u) \right\} dG(u) + \int IK_h(t-u) dF_0(u) \\ &= \int \psi_{t,h}(u) \left\{ F_0(u) - \hat{F}_n(u) \right\} d(\hat{\mathbb{G}}_n - G)(u) + \int \psi_{t,h}(u) \left\{ \delta - F_0(u) \right\} d\hat{\mathbb{P}}_n(u, \delta) \\ &\quad + \int \left\{ \psi_{t,h}(u) - \bar{\psi}_{t,h}(u) \right\} \left\{ \hat{F}_n(u) - \delta \right\} d\hat{\mathbb{P}}_n(u, \delta) + \int IK_h(t-u) dF_0(u) \\ &= \tilde{F}_{nh}^{(toy)*}(t) + \int \psi_{t,h}(u) \left\{ F_0(u) - \hat{F}_n(u) \right\} d(\hat{\mathbb{G}}_n - G)(u, \delta) \\ &\quad + \int \left\{ \psi_{t,h}(u) - \bar{\psi}_{t,h}(u) \right\} \left\{ \hat{F}_n(u) - \delta \right\} d\hat{\mathbb{P}}_n(u, \delta) \\ &= \tilde{F}_{nh}^{(toy)*}(t) + A_I + A_{II}. \end{aligned}$$

We first evaluate A_I and show that this term is $o_{P_M}(n^{-2/5})$ in probability, we have:

$$\begin{aligned} A_I &= \int \psi_{t,h}(u) \left\{ F_0(u) - \hat{F}_n(u) \right\} d(\hat{\mathbb{G}}_n - G)(u, \delta) \\ &= \int \psi_{t,h}(u) \left\{ F_0(u) - \hat{F}_n(u) \right\} d(\hat{\mathbb{G}}_n - \mathbb{G}_n)(u, \delta) \\ &\quad + \int \psi_{t,h}(u) \left\{ F_0(u) - \hat{F}_n(u) \right\} d(\mathbb{G}_n - G)(u, \delta) \end{aligned}$$

An argument similar to that of Lemma A.7 in [14] shows that

$$\int \psi_{t,h}(u) \left\{ F_0(u) - \hat{F}_n(u) \right\} d(\mathbb{G}_n - G)(u, \delta) = o_p(n^{-2/5}),$$

and hence,

$$\int \psi_{t,h}(u) \left\{ F_0(u) - \hat{F}_n(u) \right\} d(\mathbb{G}_n - G)(u, \delta) = o_{P_M}(n^{-2/5}),$$

in probability. Similarly to the proof of Lemma A.7 in [14], we can also show that

$$\int \psi_{t,h}(u) \left\{ F_0(u) - \hat{F}_n(u) \right\} d(\hat{\mathbb{G}}_n - \mathbb{G}_n)(u, \delta) = o_{P_M}(n^{-2/5}), \quad (6.15)$$

in probability, such that,

$$A_I = o_{P_M}(n^{-2/5}) \quad \text{in probability.}$$

We now study the term A_{II} . Using the same inequality for $\psi_{t,h} - \bar{\psi}_{t,h}$ as used in the second display after (11.49) on p. 333 of [15], we get for some constant $C > 0$ that:

$$|\bar{\psi}_{t,h}(u) - \psi_{t,h}(u)| \leq Ch^{-2} \left| \hat{F}_n(u) - F_0(u) \right| \quad (6.16)$$

for all u such that f_0 is positive and continuous in a neighborhood around u . We decompose the term A_{II} as follows,

$$\begin{aligned} A_{II} &= \int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \left\{ \hat{F}_n(u) - F_0(u) \right\} d\hat{\mathbb{P}}_n(u, \delta) \\ &\quad + \int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \left\{ F_0(u) - \delta \right\} d\hat{\mathbb{P}}_n(u, \delta). \end{aligned} \quad (6.17)$$

For the first term on the right-hand side of the above display we write,

$$\begin{aligned} &\int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \left\{ \hat{F}_n(u) - F_0(u) \right\} d\hat{\mathbb{P}}_n(u, \delta) \\ &= \int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \left\{ \hat{F}_n(u) - F_0(u) \right\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \\ &\quad + \int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \left\{ \hat{F}_n(u) - F_0(u) \right\} d\mathbb{P}_n(u, \delta) \\ &\leq \int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \left\{ \hat{F}_n(u) - F_0(u) \right\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \\ &\quad + Ch^{-2} \int_{t-h}^{t+h} \left\{ \hat{F}_n(u) - F_0(u) \right\}^2 d\mathbb{P}_n(u, \delta), \end{aligned} \quad (6.18)$$

where we use (6.16) in the last inequality. The first term in the display above is $o_{P_M}(n^{-2/5})$ in probability by (6.15) and (6.16). Since

$$E_{M|Z} \left\{ \hat{F}_n(t) - F_0(t) \right\}^2 < Kn^{-2/3} \quad \forall t \in (0, R),$$

in probability, we have by Markov's inequality and Fubini's theorem that,

$$\int_{t-h}^{t+h} \left\{ \hat{F}_n(u) - F_0(u) \right\}^2 d\mathbb{P}_n(u, \delta) = O_{P_M} \left(hn^{-2/3} \right) \text{ in probability.} \quad (6.19)$$

Hence, for $h \asymp n^{-1/5}$, we get for the second term in (6.18):

$$\begin{aligned} Ch^{-2} \int_{t-h}^{t+h} \left\{ \hat{F}_n(u) - F_0(u) \right\}^2 d\mathbb{P}_n(u, \delta) \\ = O_{P_M} \left(h^{-1} n^{-2/3} \right) = O_{P_M} \left(n^{-7/15} \right) = o_{P_M} \left(n^{-2/5} \right), \end{aligned}$$

in probability. For the second term of (6.17) we have,

$$\begin{aligned} \int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \{F_0(u) - \delta\} d\hat{\mathbb{P}}_n(u, \delta) \\ = \int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \{F_0(u) - \delta\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) \\ + \int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \{F_0(u) - \delta\} d(\mathbb{P}_n - P)(u, \delta). \end{aligned}$$

Similar to the arguments used in the treatment of term A_I above, we get by using again arguments similar to that of Lemma A.7 in [14] that:

$$\int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \{F_0(u) - \delta\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(u, \delta) = o_{P_M}(n^{-2/5}),$$

and

$$\int \left\{ \bar{\psi}_{t,h}(u) - \psi_{t,h}(u) \right\} \{F_0(u) - \delta\} d(\mathbb{P}_n - P)(u, \delta) = o_{P_M}(n^{-2/5}),$$

in probability. \square

6.3. The current status linear regression model: bootstrap validity

In this section we give a road map for the proof of the bootstrap validity in the current status linear regression model. We assume that the assumptions stated in Theorem 4.1 of [13] hold. Since the proof is very similar to the proof of Theorem 4.1 in [13], we leave the details to the interested reader. Consider the bootstrap score function

$$\hat{\psi}_n^{(\epsilon)}(\beta) = \int_{\hat{F}_{n,\beta}(t-\beta'x) \in [\epsilon, 1-\epsilon]} x \{ \hat{F}_{n,\beta}(t-\beta'x) - \delta \} d\hat{\mathbb{P}}_n(t, x, \delta), \quad (6.20)$$

for some fixed truncation parameter $\epsilon \in (0, 1/2)$.

The main idea is to show that

$$\begin{aligned} \hat{\psi}_n^{(\epsilon)}(\hat{\beta}_n) &= V(\hat{\beta}_n - \beta_0) + \int_{F_0(t-\beta'_0x) \in [\epsilon, 1-\epsilon]} \{x - E(X|T - \beta'_0X = t - \beta'_0x)\} \\ &\quad \cdot \{F_0(t - \beta'_0x) - \delta\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(t, x, \delta) \\ &+ \int_{F_0(t-\beta'_0x) \in [\epsilon, 1-\epsilon]} \{x - E(X|T - \beta'_0X = t - \beta'_0x)\} \\ &\quad \cdot \{F_0(t - \beta'_0x) - \delta\} d(\mathbb{P}_n - P)(t, x, \delta) \\ &+ o_{P_M}(n^{-1/2} + (\hat{\beta}_n - \beta_0)), \end{aligned} \quad (6.21)$$

in probability, where E denotes the unconditional expectation. Since

$$\hat{\psi}_n^{(\epsilon)}(\hat{\beta}_n) = 0$$

by definition of $\hat{\beta}_n$ and since by the proof of Theorem 4.1 in [13],

$$\begin{aligned} & -\sqrt{n}V(\beta_n - \beta_0) \\ &= \sqrt{n} \int_{F_0(t - \beta'_0 x) \in [\epsilon, 1 - \epsilon]} \{x - E(X|T - \beta'_0 X = t - \beta'_0 x)\} \\ & \quad \cdot \{F_0(t - \beta'_0 x) - \delta\} d(\mathbb{P}_n - P)(t, x, \delta) \\ & \quad + o_p(1 + \sqrt{n}(\beta_n - \beta_0)), \end{aligned}$$

We get that,

$$\begin{aligned} & -\sqrt{n}V(\hat{\beta}_n - \beta_n) \\ &= \sqrt{n} \int_{F_0(t - \beta'_0 x) \in [\epsilon, 1 - \epsilon]} \{x - E(X|T - \beta'_0 X = t - \beta'_0 x)\} \\ & \quad \cdot \{F_0(t - \beta'_0 x) - \delta\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(t, x, \delta) \\ & \quad + o_{P_M}(1 + \sqrt{n}(\hat{\beta}_n - \beta_n)) \end{aligned}$$

The validity of the bootstrap then follows by the arguments given in Section 4.2. Very important in the proof of (6.21) is the conditional bootstrapped L_2 -result,

$$\int \left\{ \hat{F}_n(u) - F_0(u) \right\}^2 du = O_{P_M} \left(n^{-2/3} \right) \quad \text{in probability,} \quad (6.22)$$

and the Equicontinuity lemma given on p.150 in [23].

Let $\bar{\phi}_{\hat{\beta}_n, \hat{F}_n, \hat{\beta}_n}$ be a (random) piecewise constant version of $\phi_{\hat{\beta}_n}$, where

$$\phi_\beta \stackrel{def}{=} E \{ X | T - \beta' X = u \},$$

and where, for a piecewise constant distribution function F with finitely many jumps at $\tau_1 < \tau_2 < \dots$, the function $\bar{\phi}_{\beta, F}$ is defined in the following way.

$$\bar{\phi}_{\beta, F}(u) = \begin{cases} \phi_\beta(\tau_i), & \text{if } F_\beta(u) > F(\tau_i), u \in [\tau_i, \tau_{i+1}), \\ \phi_\beta(s), & \text{if } F_\beta(u) = F(s), \text{ for some } s \in [\tau_i, \tau_{i+1}), \\ \phi_\beta(\tau_{i+1}), & \text{if } F_\beta(u) < F(\tau_i), u \in [\tau_i, \tau_{i+1}). \end{cases} \quad (6.23)$$

Similar to the proof of Theorem 4.1 in [13], we get that

$$\|\phi_{\hat{\beta}_n}(u) - \bar{\phi}_{\hat{\beta}_n, \hat{F}_n, \hat{\beta}_n}(u)\| \leq K |\hat{F}_{n, \hat{\beta}_n}(u) - F_{\hat{\beta}_n}(u)|, \quad (6.24)$$

for some constant $K > 0$. By the definition of the MLE $\hat{F}_{n, \hat{\beta}_n}$ as the slope of the

greatest convex minorant of the corresponding cusum diagram, we can write:

$$\begin{aligned}
& \hat{\psi}_n^{(\epsilon)}(\hat{\beta}_n) \\
&= \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \{x - \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x)\} \{\hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta\} d\hat{\mathbb{P}}_n(t, x, \delta) \\
&\quad + \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \{\phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x) - \bar{\phi}_{\hat{\beta}_n, \hat{F}_{n,\hat{\beta}_n}}(t - \hat{\beta}'_n x)\} \\
&\quad \quad \quad \cdot \{\hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta\} d\hat{\mathbb{P}}_n(t, x, \delta) \\
&= I + II,
\end{aligned}$$

For the second term, we have:

$$\begin{aligned}
II &= \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \left\{ \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x) - \bar{\phi}_{\hat{\beta}_n, \hat{F}_{n,\hat{\beta}_n}}(t - \hat{\beta}'_n x) \right\} \\
&\quad \quad \quad \cdot \left\{ \hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta \right\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(t, x, \delta) \\
&\quad + \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \left\{ \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x) - \bar{\phi}_{\hat{\beta}_n, \hat{F}_{n,\hat{\beta}_n}}(t - \hat{\beta}'_n x) \right\} \\
&\quad \quad \quad \cdot \left\{ \hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta \right\} d\mathbb{P}_n(t, x, \delta) \\
&= II_a + II_b
\end{aligned}$$

It is shown in the proof of Theorem 4.1 in [13] that

$$II_b = o_p(n^{-1/2} + (\hat{\beta}_n - \beta_0)),$$

and therefore

$$II_b = o_{P_M}(n^{-1/2} + (\hat{\beta}_n - \beta_0)) \text{ in probability.}$$

Using similar arguments as in the proof of Theorem 4.1 in [13] we can also show that

$$II_a = o_{P_M}(n^{-1/2}) \text{ in probability.}$$

Hence, we get:

$$\begin{aligned}
& \hat{\psi}_n^{(\epsilon)}(\hat{\beta}_n) \\
&= \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \{x - \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x)\} \{\hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta\} d\hat{\mathbb{P}}_n(t, x, \delta) \\
&\quad + o_{P_M}(n^{-1/2} + (\hat{\beta}_n - \beta_0)),
\end{aligned}$$

in probability. We now write,

$$\begin{aligned} & \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \{x - \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x)\} \{\hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta\} d\hat{\mathbb{P}}_n(t, x, \delta) \\ &= \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \{x - \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x)\} \\ & \quad \cdot \{\hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(t, x, \delta) \\ &+ \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \{x - \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x)\} \{\hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta\} d\mathbb{P}_n(t, x, \delta) \end{aligned}$$

It follows from the proof of Theorem 4.1 in [13] that there exists a random variable R_n of order $o_p(n^{-1/2} + \hat{\beta}_n - \beta_0)$ (and hence of order $o_{P_M}(n^{-1/2} + \hat{\beta}_n - \beta_0)$ in probability) such that,

$$\begin{aligned} & \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \{x - \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x)\} \{\hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta\} d\mathbb{P}_n(t, x, \delta) \\ &= \int_{F_0(t-\beta'_0 x) \in [\epsilon, 1-\epsilon]} \{x - \phi_0(t - \beta'_0 x)\} \{F_0(t - \beta'_0 x) - \delta\} d(\mathbb{P}_n - P)(t, x, \delta) \\ & \quad + \psi'_{1,\epsilon}(\beta_0)(\hat{\beta}_n - \beta_0) + R_n. \end{aligned} \tag{6.25}$$

where $\phi_0 \equiv \phi_{\beta_0}$. Therefore, (6.21) follows if we can show that,

$$\begin{aligned} & \int_{\hat{F}_{n,\hat{\beta}_n}(t-\hat{\beta}'_n x) \in [\epsilon, 1-\epsilon]} \{x - \phi_{\hat{\beta}_n}(t - \hat{\beta}'_n x)\} \{\hat{F}_{n,\hat{\beta}_n}(t - \hat{\beta}'_n x) - \delta\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(t, x, \delta) \\ &= \int_{F_0(t-\beta'_0 x) \in [\epsilon, 1-\epsilon]} \{x - \phi_0(t - \beta'_0 x)\} \{F_0(t - \beta'_0 x) - \delta\} d(\hat{\mathbb{P}}_n - \mathbb{P}_n)(t, x, \delta) \\ & \quad + o_{P_M}(n^{-1/2} + (\hat{\beta}_n - \beta_0)). \end{aligned} \tag{6.26}$$

Equality (6.26) follows by similar arguments used in the proof of (6.25) based on asymptotic equicontinuity (see Lemma 15 on p. 150 of [23]) using the closeness of $\hat{F}_{n,\hat{\beta}_n}$ to $F_{\hat{\beta}_n}$ and using entropy results for the functions $u \mapsto \hat{F}_{n,\hat{\beta}_n}(u)$ and the simpler parametric functions $u \mapsto F_{\hat{\beta}_n}(u)$ and $u \mapsto \phi_{\hat{\beta}_n}(u)$, parametrized by the finite dimensional parameter $\hat{\beta}_n$.

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