Computing Chernoff's distribution

Piet Groeneboom[1] and Jon A. Wellner[2]

November 3, 1999

Abstract

A distribution which arises in problems of estimation of monotone functions is that of the location of the maximum of two-sided Brownian motion minus a parabola. Using results of Groeneboom (1985), (1989), we present algorithms and programs for computation of this distribution and its quantiles. We also present some comparisons with earlier computations (Dykstra and Carolan (1996)) and simulations (Narayanan and Sager (1989), and Keiding, Begtrup, Scheike, and Hasibeder (1996)).

[1] Research supported in part by National Science Foundation grant DMS-9743445

[2] Research supported in part by National Science Foundation grant DMS-Research supported in part by National Science Foundation grant DMS-9532039 and NIAID grant 2R01 AI291968-04.

AMS 1980 *subject classifications.* Primary: 62E17; secondary 65D20.

Key words and phrases. Airy functions, Brownian motion, density, distribution function, monotone, parabolic drift, quadratic drift, quantiles.

1. Introduction

Our goal here is to compute, table, and plot the density, distribution function, moments, and quantiles of the location Z of the maximum of two-sided Brownian motion B minus the parabola t^2. We also provide several examples of the application of this distribution to problems including interval censoring, monotone density and hazard estimation, deconvolution, least median of squares estimation and mode estimation.

To be explicit, let $B(t)$, $-\infty < t < \infty$, be two-sided standard Brownian motion with $B(0) = 0$. Then

$$Z \equiv \mathrm{argmax}_t (B(t) - t^2)\,.$$

It follows from Lemma 2.6 of Kim and Pollard (1990) that Z is uniquely defined with probability 1. The distribution of Z apparently first arose in work of Chernoff (1964) on the estimation of the mode of a distribution function, and hence we refer to the distribution of Z as *Chernoff's distribution.*

Prakasa Rao (1969) showed that the distribution of the slope at zero of the greatest convex minorant of $B(t) + t^2$ is exactly $2Z$. This follows from the "switching relation"; see Groeneboom (1985), (2.2), page 541 for the finite sample version of this relation. Groeneboom (1985), (1989) completely described the distribution of Z and characterized analytically the *process* $\{V(a) : a \in I\!R\}$, where

$$V(a) \equiv \sup\{t \in I\!R : B(t) - (t-a)^2 \text{ is maximal.}$$

In particular, Z has a density f_Z with respect to Lebesgue measure on $I\!R$ which is symmetric about zero, and which satisfies

$$f_Z(z) \sim \tfrac{1}{2} \frac{4^{4/3}|z|}{Ai'(\tilde{a}_1)} \exp(-\tfrac{2}{3}|z|^3 + 2^{1/3}\tilde{a}_1|z|) \qquad \text{as} \qquad z \to \infty\,.$$

where $\tilde{a}_1 \approx -2.3381$ is the largest zero of the Airy function Ai and where $Ai'(\tilde{a}_1) \approx 0.7022$. The link of the distribution of Z with Airy functions is also given in Daniels and Skyrme (1985), but the process $\{V(a) : a \in I\!R\}$ is not discussed in that paper. In unpublished notes, Groeneboom and Sommeijer (1984) numerically computed the absolute moments $E(|Z|^k)$, $k = 1, \ldots, 4$. The first of these was reported by Devroye and Gyorfi (1985), page 214; and all four of them were reported by Keiding, Begtrup, Scheike, and Hasibeder (1996). Note that by symmetry of f_Z it follows that $E(Z^k) = 0$ for k odd.

2. Applications and Examples.

Here we present several examples showing how the distribution of Z enters.

Example 1. (Decreasing densities) A classical example of an application of the theory is the *Grenander estimator* of a decreasing density on $[0, \infty)$ Suppose that $X_1, \ldots, X_n$ is a sample, generated by a decreasing density f on $[0, \infty)$ that has a nonzero derivative $f'(x)$ at a point $x \in (0, \infty)$. Let $\hat{f}_n$ be the maximum likelihood estimator of f under the monotonicity restriction.

Then $\hat{f}_n$ is the left-continuous derivative of the concave majorant of the empirical distribution function, see Grenander (1956) and Groeneboom and Lopuhaä (1993). This estimator has, after "cube root n" standardization, the following limiting distribution

$$n^{1/3}\left\{\tfrac{1}{2}f(x)f'(x)\right\}^{-1/3}\left\{\hat{f}_n(x)-f(x)\right\}\xrightarrow{\mathcal{D}}2Z,\;n\to\infty,$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution, see Prakasa Rao (1969) and (for a different shorter proof) Groeneboom (1985).

Example 2. (Monotone failure rates) For this situation a similar result holds as in Example 1. For example, let $X_1,\ldots,X_n$ is a sample, generated by a distribution with an increasing failure rate r on $[0,\infty)$ and let $\hat{r}_n$ be the NPMLE of r in the class of distributions with increasing failure rate. Then, under some regularity conditions:

$$n^{1/3}\left\{\tfrac{1}{2}r(x)^2r'(x)\right\}^{-1/3}\left\{\hat{r}_n(x)-r(x)\right\}\xrightarrow{\mathcal{D}}2Z,\;n\to\infty,$$

see Prakasa Rao (1970).

Example 3. (Least median of squares estimator) Let $X_1,\ldots,X_n$ be a sample from a density strongly unimodal distribution with density f_θ, given by

$$f_\theta(x)=f(x-\theta),\;x\in\mathbb{R},$$

where the density f is differentiable and symmetric around zero. Rousseeuw (1984) introduces the *least median of squares* estimator $\hat{\theta}_n$ for the shift parameter θ and gives a heuristic argument for the result that

$$n^{1/3}\left\{\hat{\theta}_n-\theta\right\},$$

converges in distribution to $c\cdot Z$, where $c>0$ is a constant depending on the distribution function F corresponding to the density f, the density f itself and its derivative f'. A proof of this result is given in Kim and Pollard (1990). In Rousseeuw (1984) also recommendations for confidence intervals can be found.

Example 4. (Interval censoring, case 1, also called "current status model"). Suppose that X is a "survival time" with distribution function F on $[0,\infty)$ and Y is an observation time which is independent of X and has distribution function G on $[0,\infty)$. However we can observe only $(Y,1_{\{X\le Y\}})=(Y,\Delta)$, and want to estimate F, the distribution function of X based on i.i.d. replications $(Y_1,\Delta_1),\ldots,(Y_n,\Delta_n)$ of (Y,Δ). In this case the NPMLE $\widehat{\mathbb{F}}_n$ of F is known from Ayer, Brunk, Ewing, Reid, and Silverman (1955). It was proved in Groeneboom and Wellner (1992) that if F has density f and G has density g at $t_0\in(0,\infty)$ with $g(t_0)>0$, $f(t_0)>0$, then

$$n^{1/3}(\widehat{\mathbb{F}}_n(t_0)-F(t_0))\xrightarrow{\mathcal{D}}\left\{\frac{1}{2}F(t_0)(1-F(t_0))f(t_0)/g(t_0)\right\}^{1/3}2Z,\;n\to\infty.$$

Thus from Table 2 in section 2 below, it follows that an asymptotic 95% confidence interval for $F(t_0)$ is given by

$$\widehat{\mathbb{F}}_n(t_0) \pm n^{-1/3} \left\{ \frac{1}{2} \widehat{\mathbb{F}}_n(t_0)(1 - \widehat{\mathbb{F}}_n(t_0)) \widehat{f}(t_0)/\widehat{g}(t_0) \right\}^{1/3} 2 \cdot (.99818)$$

where $\widehat{f}(t_0)$ and $\widehat{g}(t_0)$ are any consistent estimators of $f(t_0)$ and $g(t_0)$ respectively; e.g. based on kernel smoothing of $\widehat{\mathbb{F}}_n$ and $\widehat{\mathbb{G}}_n(t) \equiv n^{-1} \sum_{i=1}^n 1_{\{Y_i \le t\}}$.

In the particular application discussed by Keiding et al. (1996), X_i represents "age of immunization" of individual i against rubella, Y_i represents "current age" of person i.

Example 5. (Interval censoring, case 2). In this case the data consist of a sample of observations

$$(U_i, V_i, \Delta_i, \Gamma_i),\ i = 1, \dots, n,$$

where $U_i < V_i$ and $[U_i, V_i]$ is an "observation interval" for the (hidden and unobservable) variable X_i. The variables Δ_i and Γ_i are indicators, telling us whether X_i is left of U_i, between U_i and V_i, or right of V_i:

$$\Delta_i = 1_{\{X_i \le U_i\}}, \quad \Gamma_i = 1_{\{U_i < X_i \le V_i\}}$$

The X_i are assumed to be independent of the (U_i, V_i). For analyses of this model, see, for example, Groeneboom (1996) and Geskus and Groeneboom (1999). In this situation the NPMLE $\widehat{\mathbb{F}}_n$ has to be computed by an iterative method. A fast iterative algorithm is available, the so-called *iterative convex minorant algorithm*, proposed in Groeneboom and Wellner (1992) and further analyzed in Jongbloed (1998b).

Assume that (U_i, V_i) has a density h w.r.t. Lebesgue measure, with first and second marginal densities h_1 and h_2, respectively. Moreover, suppose that F is the distribution function of the variable X_i with a density f w.r.t. Lebesgue measure, and let k_i, $i = 1, 2$, and the function a be defined by

$$k_1(u) = \int_u^\infty \frac{h(u,v)}{F(v) - F(u)}\, dv, \quad k_2(v) = \int_0^v \frac{h(u,v)}{F(v) - F(u)}\, du,$$

and

$$a(t) = \frac{h_1(t_0)}{F(t_0)} + k_1(t_0) + k_2(t_0) + \frac{h_2(t_0)}{1 - F(t_0)}.$$

Then we have at a point t in the interior of the support of the distribution function F under some regularity conditions, in particular assuming that the observation points U_i and V_i are strictly separated (i.e., $P\{V_i - U_i < \epsilon\} = 0$ for some $\epsilon > 0$) and that $f(t) > 0$:

$$n^{1/3} \{2a(t)/f(t)\}^{1/3} \{\widehat{\mathbb{F}}_n(t) - F(t)\} \xrightarrow{\mathcal{D}} 2Z,\ n \to \infty,$$

see Groeneboom (1996), Theorem 4.4.

A long-standing conjecture is that in the situation where U_i and V_i are not strictly separated the rate of convergence increases to $(n \log n)^{1/3}$ and that the limiting distribution is again given by Z, see Groeneboom and Wellner (1992), p. 100, but this conjecture has at present still not been proved or disproved. Extensions to more than two observation points are possible (the situation is not very different from "case 2", since only the two observation points surrounding the hidden variable X_i will be relevant for the analysis), but we will not further discuss this here.

Example 6. (Deconvolution) Let $X_1, \ldots, X_n$ be a sample from the convolution of an unknown distribution function F, concentrated on $[0,1]$ and the uniform distribution and let $\widehat{\mathbb{F}}_n$ be the NPMLE of F. Then, if F has a positive density $f(x)$ at $x \in (0,1)$:

$$n^{1/3}\left\{\widehat{\mathbb{F}}_n(x) - F(x)\right\} \Big/ \left\{\tfrac{1}{2}F(x)(1-F(x))f(x)\right\}^{1/3} \stackrel{\mathcal{D}}{\to} 2Z,\ n \to \infty,$$

see Theorem 4.5 in van Es (1991) or Groeneboom and Wellner (1992), p. 109. A similar result for deconvolution with the exponential distribution is given in Jongbloed (1998a), where it is shown that if $X_1, \ldots, X_n$ is a sample from the convolution of an unknown distribution function F, concentrated on $[0,\infty)$ with the standard exponential distribution, the NPMLE $\hat{F}_n$ of F satisfies

$$n^{1/3}\left\{\widehat{\mathbb{F}}_n(x) - F(x)\right\} \Big/ \left\{\tfrac{1}{2}e^{-x}f(x)\right\}^{1/3} \stackrel{\mathcal{D}}{\to} 2Z,$$

if F has a positive density $f(x)$ at $x > 0$. For related material, see van Es and van Zuijlen (1996) and van Es, Jongbloed and van Zuijlen (1998).

Example 7. (Mode Estimation; Venter's estimator). Suppose that $X_1, \ldots, X_n$ are i.i.d. with unimodal density f satisfying

$$f(x) = \gamma_0 - \frac{1}{2}\gamma(x-\theta)^2 + \frac{1}{6}\gamma_3(x-\theta)^3 + o(|x-\theta|^3)\,.$$

Then as shown by Venter (1967), his estimator $\widehat{\theta}_n$ of the mode θ satisfies

$$n^{1/5}(\widehat{\theta}_n - \theta) \to_d 2^{1/3}A^{-2/3}\gamma^{-2/3}\gamma_0 Z\,.$$

Thus, if $\hat{\gamma}_0$ and $\hat{\gamma}$ are consistent estimators of γ_0 and γ respectively, then

$$\widehat{\theta}_n \pm \frac{2^{1/3}\hat{\gamma}_0}{n^{1/5}A^{2/3}\hat{\gamma}^{2/3}} \cdot (.99818)$$

yields an approximate 95% confidence interval for the mode θ.

Narayanan and Sager (1989) give several nice examples of mode estimation via both Chernoff's estimators and Venter's estimators and their (simulated) quantiles of the distribution of Z to form confidence intervals; see especially pages 46 - 50.

Example 8. (Panel Count Data). Wellner and Zhang (1998) show that a pseudo-likelihood estimator $\widehat{\Lambda}_n$ of the mean function Λ of a counting process with "panel count" data satisfies

$$n^{1/3}(\widehat{\Lambda}_n^{ps}(t) - \Lambda(t)) \stackrel{\mathcal{D}}{\to} \left\{\frac{\sigma^2(t)\Lambda'(t)}{2G'(t)}\right\}^{1/3} 2Z$$

where $G'(t) = \sum_{k=1}^{\infty} P(K=k) \sum_{j=1}^{k} G'_{k,j}(t)$. Thus if $\hat{\sigma}^2(t)$, $\widehat{\Lambda}'$, and $\widehat{G}'(t)$ are consistent estimators of $\sigma^2(t)$, $\Lambda'(t)$, and $G'(t)$ respectively, then

$$\widehat{\Lambda}_n^{ps}(t) \pm \frac{1}{n^{1/3}} \left\{ \frac{\hat{\sigma}^2(t)\widehat{\Lambda}'(t)}{2\widehat{G}'(t)} \right\}^{1/3} \cdot 2 \cdot (.99818)$$

yields an approximate 95% confidence interval for $\Lambda(t)$.

For a rather different approach to examples of the type presented here, see Politis and Romano (1994), especially their example 2.1.1, pages 2035-2036.

3. Computation of the density f_Z and distribution function F_Z.

The density f_Z can in principle be found by solving the following partial differential equation (heat equation), given in Chernoff (1964):

$$\frac{\partial}{\partial t} u(t,x) = -\tfrac{1}{2} \frac{\partial^2}{\partial x^2} u(t,x), \tag{3.1}$$

for $x \le t^2$, under the boundary conditions:

$$u(t,t^2) \stackrel{\text{def}}{=} \lim_{x \uparrow t^2} u(t,x) = 1, \qquad \lim_{x \downarrow -\infty} u(t,x) = 0,\ t \in I\!R. \tag{3.2}$$

In terms of the (smooth) solution $u(t,x)$, the density f_Z is given by

$$f_Z(t) = \tfrac{1}{2} u_2(-t) u_2(t),\ t \in I\!R, \tag{3.3}$$

where (as in Groeneboom (1985), the function u_2 is defined by

$$u_2(t) = \lim_{x \uparrow t^2} \frac{\partial}{\partial t} u(t,x),\ t \in I\!R. \tag{3.4}$$

In fact, the original computations of the density were based on a numerical solution of this differential equation (this information is based on personal communications from professors Herman Chernoff and Willem van Zwet). The trouble with this approach is the behavior of the function u_2 for negative values of t. In fact, since, by (4.25) in Groeneboom (1985),

$$u_2(t) \sim c_1 \exp\left\{ -\tfrac{2}{3}|t|^3 - c|t| \right\},\ t \to -\infty,$$

where $c \approx 2.9458\ldots$ and $c_1 \approx 2.2638\ldots$, the function u_2 tends to zero extremely rapidly, as t decreases away from zero. Some experiments with the numerical approach by the first author, in cooperation with B. Sommeyer, back in 1984, showed that this simple analytic fact invalidates any direct numerical approach, based on the partial differential equation: an error analysis showed that even with very fine grids the numerical solution was highly unstable. For this reason a more thorough analytic analysis of the problem was made, and the results of this analysis are given in Groeneboom (1985) and Groeneboom (1989)

The following development is from Groeneboom (1985), section 4, pages 548 - 553. Define a function $p : [0, \infty) \rightarrow R$ as follows:

$$p(y) = \begin{cases} -\sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} a_k y^{3k} + \sum_{k=1}^{\infty} b_k y^{3(k-1/2)} , & \text{if } \ y \in [0,1] \\ -y^{-3/2} + 2\sqrt{2\pi} \exp(-y^3/6) \sum_{k=1}^{\infty} \exp(2^{1/3} \tilde{a}_k y) , & \text{if } \ y \in (1, \infty) . \end{cases} \tag{3.5}$$

Here the $\tilde{a_k}$'s are the zeros of the Airy function Ai, and a_k, b_k are defined recursively as follows: set $c_0 = 1$ and

$$c_n = -2^{-4} \frac{(2n-3)(2n+1)}{n^2(2n-1)} c_{n-1} , \qquad n = 1, 2, \ldots .$$

The recursive relations for the coefficients a_k and b_k follow from the integral equation (4.14) in Groeneboom (1985). The integral equation leads to an accurate and useful analytic representation of the density in a neighborhood of zero, whereas the expansion on the second line of (3.5) does similar job away from zero.

Then with $a_0 = 1$, $b_1 = 2/3$, and $B(p,q) \equiv \Gamma(p)\Gamma(q)/\Gamma(p+q)$, the standard Beta function, set

$$a_n = c_n - \sum_{k=0}^{n-1} \frac{1}{\pi k!(-2)^k} b_{n-k} B(3n-2k-1/2, k+3/2) , \qquad n = 1, 2, \ldots ; \tag{3.6}$$

$$b_n = \sum_{k=0}^{n-1} \frac{1}{k!(-2)^{k+1}} a_{n-k-1} B(3n-2k-2, k+3/2) , \qquad n = 2, 3, \ldots . \tag{3.7}$$

The reason for treating the intervals $[0, 1]$ separately is that the series using the zeros of the Airy function diverges at zero and gives a bad approximation in neighborhoods of zero.

We then define $g : I\!R \rightarrow I\!R$ by

$$\begin{aligned} g(x) \quad = \quad & 2x - \frac{1}{\sqrt{2\pi}} \int_0^\infty p(y) \ \exp(-\tfrac{1}{2} y (2x+y)^2) dy \\ & + 2\sqrt{\frac{2}{\pi}} \int_0^\infty \{(2x+y^2)y^2 + \tfrac{1}{2}(2x+y^2)^2\} \exp(-\tfrac{1}{2} y^2 (2x+y^2)^2) \, dy \end{aligned} \tag{3.8}$$

if $x \in [-1, \infty)$, and

$$g(x) = \exp(\tfrac{2}{3} x^3) 4^{1/3} \sum_{k=1}^{\infty} \exp(-2^{1/3} \tilde{a}_k x) / Ai'(\tilde{a}_k) \tag{3.9}$$

if $x \in (-\infty, -1]$; here Ai' is the derivative of the Airy function Ai. The reason for using y^2 in the integrand of the first part of the definition of g instead of y as in Groeneboom (1985), is purely numerical: the present change of variables avoids a factor of $\sqrt{y}$ in the denominator of the integrand.

Finally, the density f_Z is expressed in terms of g as

$$f_Z(z) = \tfrac{1}{2} g(z) g(-z), \qquad z \in (-\infty, \infty) . \tag{3.10}$$

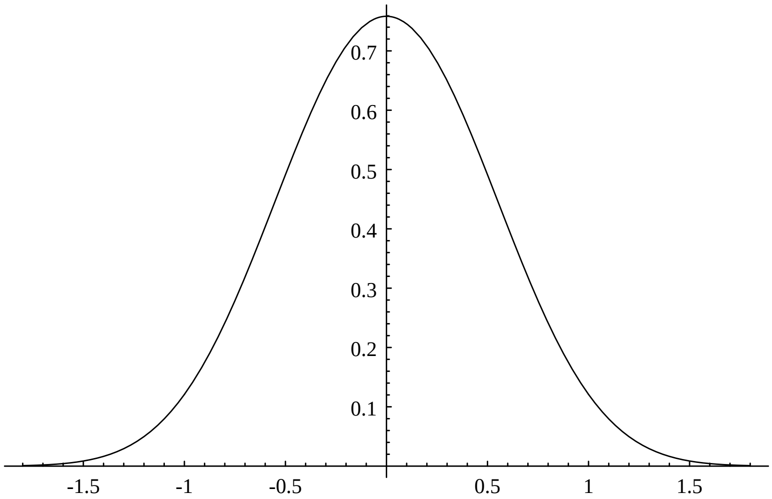

Figure 1: Density function of Z, f_Z.

The distribution function F_Z of Z is simply

$$F_Z(z) = \int_{-\infty}^{z} f_Z(w)dw = \tfrac{1}{2}\int_{-\infty}^{z} g(w)g(-w)dw\,, \qquad z \in (-\infty, \infty)\,.$$

Because of the symmetry of f_Z about 0, it suffices to calculate

$$F_Z(z) - F_Z(0) = \int_{0}^{z} f_Z(w)dw = \tfrac{1}{2}\int_{0}^{z} g(w)g(-w)dw\,, \qquad z \in [0, \infty)\,. \tag{3.11}$$

Figures 1 and 2 show plots of the density function f_Z and the distribution function F_Z respectively; in these figures we used the first 20 terms of the series defining the function p, and also the first 20 terms of the series defining g in the region $(-\infty, -1)$. The figures shown here were produced by *Mathematica*; see Wolfram (1996).

The tables were also first computed in Mathematica. Subsequently, a computer program, written in C, was developed, using some routines for computing integrals and (zeros of) Airy functions in Lau (1995). These C routines are translations into C of the routines in the NUMAL library of the ALGOL 60 routines, developed at the Mathematical Centre, Amsterdam, see Hemker (1980). The results of the C program correspond in all decimals shown with the results obtained in Mathematica, except in a few cases where Mathematica could not reach sufficient accuracy (giving small differences in the last decimals). The C program was originally written on a Macintosh powerbook, using the Metrowerks Code Warrior C compiler and the sources could be compiled without any change on an HP Unix workstation by the standard C compiler available on this workstation. In the demonstration version of the program Table 1 of the present paper is computed.

We also tried to provide a Microsoft Windows 98 executable, but in this case ran into trouble, since compilation by the Microsoft Visual C++ compiler, version 6.0, produced a so-called “release version” that either did not work or produced the wrong results (interestingly, depending on the

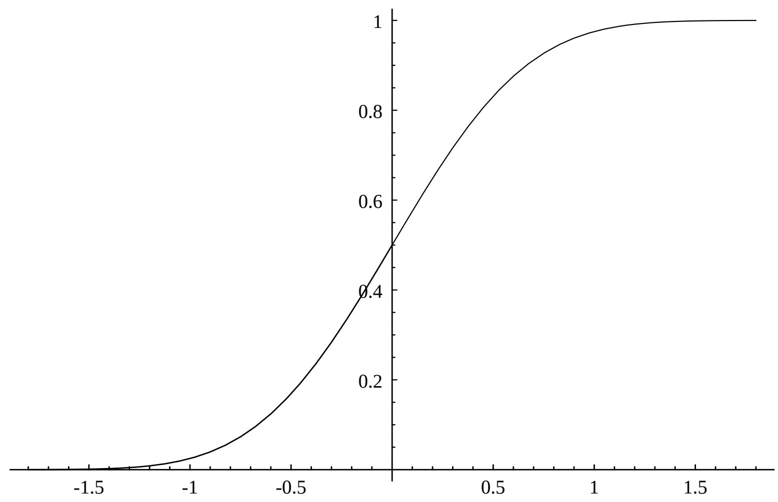

Figure 2: Distribution function of Z, F_Z.

machine on which it was run!). The so-called "debug version" worked o.k., but was very slow. Consulting the available information on differences between "debug" and "release" versions did not help us to solve the problem. Disabling the "maximize speed" option in compiling the release version also produced a correctly but slowly working version! So, in view of these experiences, we recommend compiling the C sources either on a machine using the Unix operating system or on a Macintosh power PC, using Metrowerks Code Warrior. Compilation using Microsoft Visual C++ may produce unexpected results!

Dykstra and Carolan (1997) computed the density function f_Z by numerical Fourier inversion of formula (3.8), page 91, Groeneboom (1989). This section showed how f_Z is computable without numerical Fourier inversion.

Table 1 gives the distribution function $F_Z(z)$ and the density function $f_Z(z)$ for $z = 0.0(.01)2.0$. We took $n = 20$ in our computation of the power series; the demonstration program allows the user take a different number of terms before starting the computation. Our experience is that in going beyond 20 (in choosing the number of terms in the power series), the results did not change in 9 decimals. The C code also contains routines for computing moments and quantiles; these routines can all be found in the source "main.c". We used these routines in producing the other tables below.

Table 1. Values of the distribution function F_Z and density function f_Z.

z	$F_Z(z)$	$f_Z(z)$	z	$F_Z(z)$	$f_Z(z)$	z	$F_Z(z)$	$f_Z(z)$
.00	.500000	0.758345	.30	.716352	0.649874	.60	.875858	.403594
.01	.507583	0.758215	.31	.722817	0.643059	.61	.879851	.394887
.02	.515163	0.757828	.32	.729213	0.636088	.62	.883756	.386214
.03	.522739	0.757183	.33	.735538	0.628967	.63	.887575	.377580
.04	.530306	0.756281	.34	.741792	0.621704	.64	.891308	.368989
.05	.537863	0.755123	.35	.747972	0.614303	.65	.894955	.360447
.06	.545408	0.753709	.36	.754077	0.606771	.66	.898517	.351960
.07	.552937	0.752042	.37	.760107	0.599115	.67	.901994	.343531
.08	.560448	0.750122	.38	.766059	0.591341	.68	.905388	.335166
.09	.567938	0.747951	.39	.771933	0.583455	.69	.908698	.326870
.10	.575406	0.745532	.40	.777728	0.575464	.70	.911925	.318646
.11	.582848	0.742866	.41	.783442	0.567374	.71	.915071	.310499
.12	.590263	0.739957	.42	.789075	0.559192	.72	.918136	.302433
.13	.597647	0.736806	.43	.794626	0.550925	.73	.921120	.294452
.14	.604998	0.733416	.44	.800094	0.542578	.74	.924025	.286560
.15	.612314	0.729792	.45	.805477	0.534159	.75	.926852	.278760
.16	.619593	0.725935	.46	.810776	0.525674	.76	.929601	.271057
.17	.626832	0.721849	.47	.815991	0.517130	.77	.932273	.263452
.18	.634029	0.717539	.48	.821119	0.508533	.78	.934870	.255950
.19	.641182	0.713008	.49	.826161	0.499890	.79	.937392	.248553
.20	.648289	0.708260	.50	.831117	0.491208	.80	.939841	.241264
.21	.655347	0.703299	.51	.835985	0.482492	.81	.942218	.234086
.22	.662354	0.698131	.52	.840766	0.473749	.82	.944523	.227020
.23	.669309	0.692758	.53	.845460	0.464986	.83	.946759	.220070
.24	.676208	0.687187	.54	.850066	0.456209	.84	.948925	.213237
.25	.683052	0.681422	.55	.854584	0.447424	.85	.951024	.206523
.26	.689836	0.675469	.56	.859014	0.438638	.86	.953056	.199931
.27	.696560	0.669332	.57	.863357	0.429855	.87	.955023	.193460
.28	.703222	0.663017	.58	.867612	0.421083	.88	.956926	.187113
.29	.709820	0.656529	.59	.871779	0.412327	.89	.958766	.180891

Table 1 continued

z	$F_Z(z)$	$f_Z(z)$	z	$F_Z(z)$	$f_Z(z)$	z	$F_Z(z)$	$f_Z(z)$
.90	.960544	.174795	1.27	.994448	.0346458	1.64	.999631	.0031329
.91	.962262	.168827	1.28	.994785	.0328273	1.65	.999662	.0028995
.92	.963921	.162985	1.29	.995105	.0310864	1.66	.999689	.0026817
.93	.965522	.157272	1.30	.995407	.0294208	1.67	.999715	.0024785
.94	.967067	.151687	1.31	.995693	.0278282	1.68	.999739	.0022891
.95	.968556	.146231	1.32	.995964	.0263065	1.69	.999761	.0021127
.96	.969992	.140904	1.33	.996220	.0248534	1.70	.999781	.0019485
.97	.971375	.135705	1.34	.996461	.0234666	1.71	.999800	.0017957
.98	.972706	.130635	1.35	.996689	.0221441	1.72	.999817	.0016537
.99	.973988	.125694	1.36	.996904	.0208837	1.73	.999833	.0015219
1.00	.975221	.120880	1.37	.997107	.0196831	1.74	.999848	.0013995
1.01	.976406	.116194	1.38	.997298	.0185404	1.75	.999861	.0012861
1.02	.977545	.111633	1.39	.997478	.0174534	1.76	.999873	.0011810
1.03	.978639	.107199	1.40	.997648	.0164201	1.77	.999885	.0010836
1.04	.979689	.102889	1.41	.997807	.0154385	1.78	.999895	.0009936
1.05	.980697	.0987031	1.42	.997957	.0145066	1.79	.999905	.0009103
1.06	.981664	.0946394	1.43	.998097	.0136226	1.80	.999913	.0008334
1.07	.982590	.0906969	1.44	.998229	.0127844	1.81	.999921	.0007625
1.08	.983478	.0868741	1.45	.998353	.0119902	1.82	.999929	.0006970
1.09	.984328	.0831697	1.46	.998469	.0112384	1.83	.999935	.0006367
1.10	.985142	.0795821	1.47	.998578	.0105269	1.84	.999941	.0005811
1.11	.985920	.0761096	1.48	.998680	.0098542	1.85	.999947	.0005300
1.12	.986665	.0727506	1.49	.998775	.0092187	1.86	.999952	.0004830
1.13	.987376	.0695033	1.50	.998864	.0086186	1.87	.999957	.0004399
1.14	.988055	.0663658	1.51	.998948	.0080523	1.88	.999961	.0004003
1.15	.988703	.0633362	1.52	.999025	.0075183	1.89	.999965	.0003639
1.16	.989322	.0604126	1.53	.999098	.0070151	1.90	.999968	.0003306
1.17	.989912	.0575930	1.54	.999166	.0065413	1.91	.999971	.0003001
1.18	.990474	.0548753	1.55	.999229	.0060955	1.92	.999974	.0002722
1.19	.991010	.0522574	1.56	.999288	.0056763	1.93	.999977	.0002468
1.20	.991520	.0497372	1.57	.999343	.0052824	1.94	.999979	.0002235
1.21	.992005	.0473125	1.58	.999394	.0049125	1.95	.999981	.0002022
1.22	.992466	.0449811	1.59	.999441	.0045655	1.96	.999983	.0001828
1.23	.992905	.0427408	1.60	.999485	.0042401	1.97	.999985	.0001652
1.24	.993321	.0405894	1.61	.999526	.0039352	1.98	.999986	.0001491
1.25	.993717	.0385246	1.62	.999564	.0036497	1.99	.999988	.0001345
1.26	.994092	.0365441	1.63	.999599	.0033826	2.00	.999989	.0001212

4. Quantiles of F_Z and some comparisons.

Dykstra and Carolan (1997) suggested that f_Z and F_Z are closely approximated by the $N(0,(.52)^2)$ density and distribution functions respectively. While this results in a simple approximation for the corresponding quantiles $F_Z^{-1}(p)$, the differences between the exact quantiles and the approximate quantiles, or exact distribution function and approximate distribution function based on the normal approximation can be substantial.

Table 2 compares a few quantiles computed directly by inverting the distribution function, computed in the preceding section, with analytically computed, approximate (by a normal distribution approximation), and Monte Carlo quantiles as computed by Dykstra and Carolan (1997), Narayanan and Sager (1989), and Keiding et al. (1996). The Dykstra and Carolan (1997) approach seems to fail in the tail, and we indeed believe that it is absolutely necessary to use different representations of the density in a neighborhood of zero and in the tail (which is a common phenomenon in the numerical evaluation of special functions), whereas Dykstra and Carolan (1997) essentially use the same representation for the whole domain. The normal approximation clearly cannot be good for the whole domain; is this case it is reasonable for the outer values, but not so good for the intermediate values. The results of the Monte Carlo simulation of Narayanan and Sager seem pretty good for the values tabulated here and slightly better than the Monte Carlo simulation results, reported by Keiding et al. Neverthless, there is no need for Monte Carlo simulations and these will generally produce results deteriorating rapidly if one goes further out in the tail.

Table 3 gives further quantiles of the distribution F_Z.

Table 2. Comparison of several computations and estimators of the quantiles $F_Z^{-1}(p)$ for certain p's.

Percentile p	Exact $F_Z^{-1}(p)$	Dykstra and Carolan; Fourier inversion	$N(0,(.52)^2)$	Narayanan and Sager; Monte-Carlo	Keiding et al. Monte-Carlo
.9	.664235	.664	.666	.658	.66
.95	.845081	.846	.856	.838	.836
.975	.998181	.998	1.018	.986	1.009
.99	1.171530	1.156	1.205	1.156	1.176
.995	1.286659	1.270	1.314	1.281	1.278
.999	1.516664	1.452	1.515	1.510	NA

Table 3. Quantiles of $F_Z^{-1}(p)$ for $p = .5(.01).99$ and $p = .99(.001).999$.

p	$F_Z^{-1}(p)$	p	$F_Z^{-1}(p)$
.50	0.00	.80	0.439828
.51	0.013187	.81	0.458525
.52	0.026383	.82	0.477804
.53	0.039595	.83	0.497731
.54	0.052830	.84	0.518383
.55	0.066096	.85	0.539855
.56	0.079402	.86	0.562252
.57	0.092757	.87	0.585706
.58	0.106168	.88	0.610378
.59	0.119645	.89	0.636468
.60	0.133196	.90	0.664235
.61	0.146831	.91	0.694004
.62	0.160560	.92	0.726216
.63	0.174393	.93	0.761477
.64	0.188342	.94	0.800658
.65	0.202418	.95	0.845081
.66	0.216633	.96	0.896904
.67	0.230999	.97	0.960057
.68	0.24553	.98	1.043030
.69	0.260242	.99	1.171530
.70	0.275151	.991	1.189813
.71	0.290274	.992	1.209897
.72	0.305629	.993	1.232241
.73	0.321238	.994	1.257496
.74	0.337123	.995	1.286659
.75	0.353308	.996	1.321370
.76	0.369821	.997	1.364637
.77	0.386694	.998	1.423026
.78	0.403959	.999	1.516664
.79	0.421656	.9999	1.784955

5. Moments of Z.

As remarked in the introduction, the first four moments were computed by Groeneboom and Sommeijer (1984). Table 3 shows the first 10 moments of Z.

Table 4. Absolute moments of Z; $E(|Z|^k)$, $k = 1(1)10$.

k	$E(\|Z\|^k)$
1	.41273655
2	.26355964
3	.21135025
4	.19715702
5	.20573334
6	.23455025
7	.28760426
8	.37509901
9	.51604236
10	.74410271

REFERENCES

Ayer, M., Brunk, H. D., Ewing, G. M., Reid, W. T., Silverman, E. (1955). An empirical distribution function for sampling with incomplete information. *Ann. Math. Statist.* **26**, 641 - 647.

Chernoff, H. (1964). Estimation of the mode. *Ann. Statist. Math.* **16**, 31 - 41.

Daniels, H. E. and Skyrme, T. H. R. (1985). The maximum of a random walk whose mean path has a maximum. *Adv. in Appl. Probab.* **17**, 85-99.

Devroye, L. and Györfi, L. (1985). *Nonparametric Density Estimation: the L_1 View.* Wiley, New York.

Dykstra, R. and Carolan, C. (1997). The distribution of the argmax of two-sided Brownian motion with quadratic drift. Technical Report 266, Department of Statistics, University of Iowa.

Geskus, R. B. and Groeneboom, P. (1999). Asymptotically optimal estimation of smooth functionals for interval censoring, case 2. *Ann. Statist.* **27**, 627-674.

Grenander, U. (1956). On the theory of mortality measurement. Part II *Scandinavisk Aktuarietidskrift* **39**, 125-153.

Groeneboom, P. and Sommeijer, B. (1984). Computation of the distribution of the location of the maximum of Brownian motion minus a parabola. *Unpublished technical report*, University of Amsterdam.

Groeneboom, P. (1985). Estimating a monotone density. In *Proceedings of the Berkeley Conference in Honor of Jerzy Neyman and Jack Kiefer, II.* L. M Le Cam and R. A. Olshen, editors, 535 - 555. Wadsworth, Belmont. Springer-Verlag, New York.

Groeneboom, P. (1989). Brownian motion with a parabolic drift and Airy functions. *Probab. Th. Rel. Fields* **81**, 79 - 109.

Groeneboom, P. and Wellner, J. A. (1992). *Information Bounds and Nonparametric Maximum Likelihood Estimation.* Birkhäuser, Basel.

Groeneboom, P. and Lopuhaä, H. P. (1993). Isotonic estimators of monotone densities and distribution functions: basic facts. *Statistica Neerlandica* **47**, 175-1833.

Groeneboom, P. (1996). *Lectures on inverse problems*, in: Lectures on probability theory, Ecole d'Eté de Probabilités de Saint-Flour XXIV-1994, Editor: P. Bernard. Springer Verlag, Berlin.

Hemker, P. W. (ed.) (1980). *NUMAL, Numerical Procedures in ALGOL 60.* MC Syllabus, 47.1-47.7, Mathematisch Centrum (now CWI), Amsterdam.

JONGBLOED, G. (1998a). Exponential deconvolution: two asymptotically equivalent estimators. *Statistica Neerlandica*, **52**, 6-17.

JONGBLOED, G. (1998b). The iterative convex minorant algorithm for nonparametric estimation *J. Comp. Graph. Statist.*, **7**, 310-321.

KEIDING, N., BEGTRUP, K., SCHEIKE, T. H., HASIBEDER, G. (1996). Estimation from current-status data in continuous time. *Lifetime Data Analysis* **2**, 119- 129.

KIM, J. AND POLLARD, D. (1990). Cube root asymptotics. *Ann. Statist.* **18**, 191 - 219.

LAU, H. T. (1995). *A numerical library in C for scientists and engineers.* CRC Press, Boca Raton.

NARAYANAN, A. AND SAGER, T. W. (1989). Table for the asymptotic distribution of univariate mode estimators. *J. Statist. Comp. Simul.* **33**, 37 - 51.

POLITIS, D. N., AND ROMANO, J. P. (1994). Large sample confidence regions based on subsamples under minimal assumptions. *Ann. Statist.* **22**, 2031 - 2050.

PRAKASA RAO, B. L. S. (1969). Estimation of a unimodal density. *Sankhya A* **31**, 23 - 36.

PRAKASA RAO, B. L. S. (1970). Estimation fr distributions with monotone failure rate. *Ann. Math. Statist.* **41**, 507 - 519.

ROUSSEEUW, P. J. (1984). Least median of squares regression. *J. Amer. Statist. Assoc.* **79**, 871-880.

TEMME, N. M. (1985). A convolution integral equation solved by Laplace transformation. *J. Comp. Appl. Math.* **13**, 609-613.

VAN ES, A. J. (1991). Aspects of nonparametric density estimation. CWI Tract **77**. Centre for Mathematics and Computer Science, Amsterdam.

VAN ES, A. J. AND VAN ZUIJLEN, M. C. A. (1996). Convex minorant estimators of distributions in nonparametric deconvolution problems. *Scand. J. Statist.* **23**, 85-104.

VAN ES, A. J., JONGBLOED, G. AND VAN ZUIJLEN, M. C. A. (1998). Isotonic inverse estimators for nonparametric deconvolution. *Ann. Statist.* **26**, 2395-2406.

VENTER, J. H. (1967). On estimation of the mode. *Ann. Math. Statist.* **38**, 1446 - 1455.

WELLNER, J. A. AND ZHANG, Y. (1998). Two estimators of the mean of a counting process with panel count data. *Technical Report* **341**, Department of Statistics, University of Washington, Seattle. `http://www.stat.washington.edu/www/research/reports/`

WOLFRAM, S. (1996). *The Mathematica Book.* Third Edition. Wolfram Research, Champaign.

Department of Mathematics
Delft University of Technology
2628 CD Delft
The Netherlands
e-mail: P.Groeneboom@its.tudelft.nl

Department of Statistics
University of Washington
P.O. Box 354322
Seattle, Washington 98195-4322
U.S.A.
e-mail: jaw@stat.washington.edu