ESTIMATE ON THE PATHWISE LYAPUNOV EXPONENT OF LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. In this paper we study the problem of estimating the pathwise Lyapunov exponent for linear stochastic systems with multiplicative noise and constant coefficients. We present a Lyapunov type matrix inequality that is closely related to this problem, and show under what conditions we can solve the matrix inequality. From this we can deduce an upper bound for the Lyapunov exponent.

In the converse direction it is shown that a necessary condition for the stochastic system to be pathwise asymptotically stable can be formulated in terms of controllability properties of the matrices involved.

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AMS 2000 Subject classification: 93E15, 15A24, 60H10

1. Introduction

Consider, in $\mathbb{R}^n$, the linear SDE

$$\begin{cases} dx(t) = Ax(t)\ dt + \sum_{i=1}^k B_i x(t)\ dW_i(t), \quad t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times n}$, $i = 1, \dots k$ and $(W_i)_{i=1}^k$ are independent standard Brownian motions in $\mathbb{R}$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. As a special case we will often consider the equation

$$dx(t) = Ax(t)\ dt + Bx(t)\ dW(t). \tag{2}$$

It is well known (see e.g. [5]) that for any choice of A, $(B_i)_{i=1}^k$ and x_0 a unique solution to (1), denoted as $x(t; x_0)$, exists.

In this paper we are interested in the stability properties of the solution of (1). More specifically, we want to estimate the *pathwise Lyapunov exponent*, defined as

$$\lambda := \limsup_{t\to\infty} \frac{1}{t} \log |x(t; x_0)|, \quad \text{a.s.} \tag{3}$$

In the case of a linear SDE with constant coefficients 'lim sup' may be replaced by 'lim' (see [2]):

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log |x(t; x_0)|, \quad \text{a.s.}$$

Some classical references on the theory of Lyapunov exponents of random dynamical systems are [1] and [11].

1.1. **History of the problem.** In 1967 Khas'minskii [7] calculates the Lyapunov exponent (theoretically) by rewriting (1) in spherical coordinates as follows. First we write (1) in Stratonovich form

$$dx(t) = \widetilde{A}x(t)\ dt + \sum_{i=1}^{k} B_i x(t) \circ dW_i(t), \tag{4}$$

with $\widetilde{A} = A - \frac{1}{2}\sum_{i=1}^k B_i^2$.

Write $y(t) := x(t)/|x(t)|$ and $\lambda(t) := \log|x(t)|$, $t \geq 0$. Then y is a process on S^{n-1}, the unit sphere of dimension $n-1$. By using Itô's formula it may be derived that the differential equations for y are autonomous: they do not depend on $\lambda(t)$. By compactness of S^{n-1} at least one invariant measure μ exists for y.

Khas'minskii then assumes a strong non-singularity condition on the (B_i), namely

$$\sum_{i=1}^{k}(B_i x)(B_i x)^T \text{ is positive definite for all } x \in \mathbb{R}^n,\ x \neq 0. \tag{5}$$

Due to this condition, $\mathbb{P}_{y_0}(y(t) \in U) > 0$ for all open $U \subset S^1$ and $(y(t))_{t\geq 0}$ is strong Feller. By an earlier theorem of Khas'minskii ([6]), μ is therefore the unique invariant measure for $(y(t))_{t\geq 0}$ on S^{n-1}, and hence it is ergodic.

By Itô's formula, the process $(\lambda(t))_{t\geq 0}$ can be shown to satisfy

$$\lambda(t) = \lambda(0) + \int_0^t \Phi(y(s))\ ds + \sum_{i=1}^{k}\int_0^t \langle B_i y(s), y(s)\rangle\ dW_i(s) \quad \text{a.s.}, \tag{6}$$

with

$$\Phi(z) := \langle Az, z\rangle + \tfrac{1}{2}\sum_{i=1}^{k}||B_i z||^2 - \sum_{i=1}^{k}\langle B_i z, z\rangle^2, \quad z \in S^{n-1}. \tag{7}$$

Now using the strong law of large numbers for martingales (see Theorem A.1 in the stochastic term in (6) dissappears and by ergodicity of μ (following from the uniqueness of μ)

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi(y(s))\ ds = \int_{S^{n-1}}\Phi(z)\ d\mu(z) \quad \text{a.s.}$$

We may conclude that

$$\lim_{t\to\infty}\frac{1}{t}\log|x(t)| = \int_{S^{n-1}}\Phi(z)\ d\mu(z) \quad \text{a.s.} \tag{8}$$

As stated above, (5) is stronger than necessary for establishing the uniqueness of the invariant measure μ on S^{n-1}. A better understanding of the structure of ergodic invariant measures on manifolds (e.g. S^{n-1}) is provided by [8].

In [10], Mao provides a way of estimating the Lyapunov exponent. In the linear case this boils down to requiring that

$$\langle Az, z\rangle \leq \alpha \quad \text{and} \quad \tfrac{1}{2}\sum_{i=1}^{k}||B_i z||^2 - \sum_{i=1}^{k}\langle B_i z, z\rangle^2 \leq \beta, \quad z \in S^{n-1} \tag{9}$$

so that $\Phi(z) \leq \alpha + \beta$, $z \in S^{n-1}$. Therefore (6) shows that

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq \alpha + \beta \quad \text{a.s.}$$

This approach can be extended using a Lyapunov function, see [11], Theorem 4.3.3. Note that in order for β to be negative (in which case the noise has a stabilizing effect), we need that B_i is non-degenerate, i.e. $\ker B_i = \{0\}$ for some i. In this paper we wish to address stabilizing effects of the noise, even when it is degenerate.

1.2. **Outline of this paper.** First we state some preliminary results on deterministic and commutative systems in Section 2. In Section 3 an upper bound for the Lyapunov exponent is obtained by studying a particular matrix inequality. This is the most important result of this paper. In Section 4 we obtain a converse result, namely a necessary condition for pathwise stability, i.e. $\lambda \leq 0$. In Appendix A the strong law of large numbers for martingales is proven.

2. Preliminary results

In this section we briefly mention some result on deterministic and commutative systems, which will be useful in the remainder of this paper.

2.1. **Deterministic systems.** For square matrices A and B, let $\mathfrak{s}(A)$ denote the *spectral bound of* A, and $\mathfrak{r}(B)$ the *spectral radius of* B, i.e.

$$\mathfrak{s}(A) := \sup\{\Re\lambda : \lambda \in \sigma(A)\} \quad \text{and} \quad \mathfrak{r}(B) := \sup\{|\lambda| : \lambda \in \sigma(B)\}.$$

Furthermore let $\omega_0(A)$ denote the growth bound of A, i.e.

$$\omega_0(A) = \inf\{\omega \in \mathbb{R} : \exists_{M \geq 1} \text{ s.t. } ||\exp(At)|| \leq M e^{\omega t} \text{ for all } t \geq 0\}.$$

By this definition the Lyapunov exponent for deterministic systems $\dot{x}(t) = Ax(t)$ is is given by $\lambda = \omega_0$.

In finite dimensions the following equalities hold:

$$\lambda = \mathfrak{s}(A) = \omega_0(A) = \frac{1}{t} \log\ \mathfrak{r}\left(\exp(At)\right), \quad \text{for all } t \geq 0. \tag{10}$$

See [4], Proposition IV.2.2 and Theorem IV.3.11.

2.2. **Commutative case.** As an appetizer, consider the particular case of (1) where A and all B_i, $i = 1, \dots, k$ commute. Then the solution is given by

$$x(t) = \exp\left[t\left(A - \tfrac{1}{2}\sum_{i=1}^k B_i^2\right) + \sum_{i=1}^k W_i(t) B_i\right] x_0 \quad \text{a.s.},$$

and

$$\frac{1}{t}\log|x(t)| \leq \frac{1}{t}\log|x_0| + \frac{1}{t}\log\left|\left|\left(\exp\left(A - \tfrac{1}{2}\sum_{i=1}^k B_i^2\right)t\right)\right|\right| + \frac{1}{t}\sum_{i=1}^k ||B_i||\ |W_i(t)|.$$

Now using the strong law of large numbers for martingales (see Theorem A.1)

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^k ||B_i||\ |W_i(t)| = 0 \quad \text{a.s.},$$

and using the observations above,

$$\lim_{t\to\infty} \frac{1}{t}\log\left|\left|\left(\exp\left(A - \tfrac{1}{2}\sum_{i=1}^k B_i^2\right)t\right)\right|\right| = \mathfrak{s}\left(A - \tfrac{1}{2}\sum_{i=1}^k B_i^2\right).$$

Hence we find

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq \mathfrak{s}\left(A - \tfrac{1}{2}\sum_{i=1}^k B_i^2\right), \quad \text{a.s.} \tag{11}$$

3. Estimating the Lyapunov exponent by means of a matrix inequality

In this section we use the particular Lyapunov function $V(x) := \langle Qx, x\rangle$ with Q positive definite, to obtain an estimate for the pathwise Lyapunov exponent for the solution x of (1). It is shown that a Lyapunov type inequality for Q can be formulated which gives a sufficient condition for x to have a particular Lyapunov exponent. In Theorem 3.6 general conditions are formulated such that a positive definite solution to the mentioned matrix inequality exists. As a corollary we formulate conditions on A and B such that the solution of (2) has a particular Lyapunov exponent in Theorem 3.7.

3.1. **Proposition.** Suppose there exists a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$ such that

$$\langle Qz, z\rangle \left[2\langle QAz, z\rangle + \sum_{i=1}^k \langle QB_i z, B_i z\rangle - 2\lambda\langle Qz, z\rangle\right] \leq 2\sum_{i=1}^k \langle Qz, B_i z\rangle^2 \quad \text{for all } z \in \mathbb{R}^n. \tag{12}$$

Then

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq \lambda \quad \text{a.s.},$$

with x the solution of (1).

Proof: If $x_0 = 0$, then $\mathbb{P}(x(t) = 0) = 1$, $t \geq 0$, and the required estimate holds trivially.

Suppose $x_0 \neq 0$. Let $y(t) := \langle Qx(t), x(t)\rangle$, $t \geq 0$. Then by uniqueness of the solution of (1) and positiveness of Q, $\mathbb{P}(y(t) = 0) = 0$ for all $t \geq 0$.

By Itô's formula and (12),

$$\begin{aligned} d\log y(t) &= \frac{1}{y(t)}\left[2\langle Qx(t), Ax(t)\rangle\ dt + 2\sum_{i=1}^k \langle Qx(t), B_i x(t)\rangle\ dW_i(t)\right] \\ &\quad - \sum_{i=1}^k \frac{2}{y(t)^2}\langle Qx(t), B_i x(t)\rangle^2\ dt + \sum_{i=1}^k \frac{1}{y(t)}\langle QBx(t), B_i x(t)\rangle\ dt \\ &= \left\{\frac{1}{y(t)}\left[2\langle QAx(t), x(t)\rangle + \sum_{i=1}^k \langle QB_i x(t), B_i x(t)\rangle\right] - \sum_{i=1}^k \frac{2}{y(t)^2}\langle Qx(t), B_i x(t)\rangle^2\right\} dt \\ &\quad + \sum_{i=1}^k \frac{2}{y(t)}\langle Qx(t), B_i x(t)\rangle\ dW_i(t) \\ &\leq 2\lambda\ dt + \sum_{i=1}^k \frac{2}{y(t)}\langle Qx(t), B_i x(t)\rangle\ dW_i(t) \quad \text{a.s.} \end{aligned}$$

Now by boundedness of $\frac{\langle Qx(t), B_i x(t)\rangle}{\langle Qx(t), x(t)\rangle}$, $i = 1, \ldots, k$ and the law of large numbers for martingales (Theorem A.1)

$$\frac{1}{t}\int_0^t \frac{2\langle Qx(s), B_i x(s)\rangle}{y(s)}\ dW_i(s) \to 0 \quad (t\to\infty), \quad \text{a.s.}, \quad i = 1, \ldots, k,$$

so

$$\limsup_{t\to\infty} \frac{1}{t}\log y(t) \leq 2\lambda \quad \text{a.s.},$$

and by positiveness of Q, this implies

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq \lambda \quad \text{a.s.}$$

□

3.2. **Remark.** Recall the definition of Φ in (7). If the conditions of Proposition 3.1 hold and we equip $\mathbb{R}^n$ with the inner product

$$\langle x, y\rangle_Q := \langle Qx, y\rangle, \quad x, y \in \mathbb{R}^n,$$

then we see that (12) is equivalent to stating that

$$\Phi_Q(z) := \langle Az, z\rangle_Q + \tfrac{1}{2}\sum_{i=1}^{k}\langle B_i z, B_i z\rangle_Q - \sum_{i=1}^{k}\langle B_i z, z\rangle_Q^2 \le \lambda$$

for $z \in S_Q^{n-1} := \{x \in \mathbb{R}^n : \langle x, x\rangle_Q = 1\}$, the unit sphere corresponding to the inner product $\langle \cdot,\cdot\rangle_Q$. So contrary to the setting of Section 1.1, we do not require a unique invariant measure on S_Q^{n-1}, since $\Phi_Q(\cdot) \le \lambda$ on the entire Q-unit sphere anyway.

3.3. **Example.** One might ask whether it is at all possible that $\Phi_Q \le \lambda$ on S_Q^{n-1} for some but not all positive definite matrices Q. Already in the deterministic case this can be seen in the next example, phrased as a proposition:

3.4. **Proposition.** Let $A = \begin{bmatrix}\alpha_1 & 1\\ 0 & \alpha_2\end{bmatrix}$ with $\alpha_1, \alpha_2 \in \mathbb{R}$ eigenvalues of A, and let $Q := \begin{bmatrix}1 & 0\\ 0 & q\end{bmatrix}$ with $q > 0$. Then for any $\lambda > \max(\alpha_1, \alpha_2)$ there exists a q such that

$$\langle QAx, x\rangle \le \lambda\langle Qx, x\rangle, \quad x \in \mathbb{R}^2,$$

and consequently

$$A^*Q + QA - 2\lambda Q \le 0.$$

Here, and in the following, we write "$\le$" for the partial order induced by the positive cone consisting of positive semidefinite matrices.

Proof: For fixed $\lambda \in \mathbb{R}$ and $q > 0$ we calculate

$$\langle QAx, x\rangle - \lambda\langle Qx, x\rangle = (\alpha_1 - \lambda)x_1^2 + q(\alpha_2 - \lambda)x_2^2 + x_1x_2.$$

Now since for any $\gamma \in \mathbb{R}$

$$|x_1x_2| = \left|\frac{1}{\gamma}x_1\gamma x_2\right| \le \left|\frac{1}{\gamma}x_1\right|\;|\gamma x_2| \le \frac{1}{2\gamma^2}x_1^2 + \frac{\gamma^2}{2}x_2^2$$

holds, we have

$$\langle QAx, x\rangle - \lambda\langle Qx, x\rangle \le \left(\alpha_1 - \lambda + \frac{1}{2\gamma^2}\right)x_1^2 + (q(\alpha_2 - \lambda) + \frac{\gamma^2}{2})x_2^2,$$

which is equal to zero for any x_1, x_2 if

$$\alpha_1 - \lambda + \frac{1}{2\gamma^2} = 0 \quad \text{and} \quad q(\alpha_2 - \lambda) + \frac{\gamma^2}{2} = 0.$$

This is satisfied for

$$\gamma^2 = 2q(\lambda - \alpha_2) \quad \text{and} \quad \lambda = \frac{\alpha_1 + \alpha_2}{2} + \frac{1}{2}\sqrt{(\alpha_1 - \alpha_2)^2 + \frac{1}{q}}.$$

When $q \to \infty$ then

$$\lambda \to \frac{\alpha_1 + \alpha_2}{2} + \frac{|\alpha_1 - \alpha_2|}{2} = \max(\alpha_1, \alpha_2).$$

□

A counterexample to show that the choice of q matters is given by $\alpha_1 = \alpha_2 = \alpha$, $q = 1$, $x_1 = x_2 = \frac{1}{2}\sqrt{2}$. Then

$$\langle QAx, x\rangle = \alpha + \tfrac{1}{2} > \max(\alpha_1, \alpha_2)|x|^2.$$

In the remainder of this section we will establish conditions such that Q and λ exist as required in Proposition 3.1.

3.5. **Lemma.** Suppose there exists a positive definite matrix $Q \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$ and constants $b_i \in \mathbb{R}$, $i = 1, \ldots, k$ such that

$$\left(A + \sum_{i=1}^k b_i B_i\right)^* Q + Q\left(A + \sum_{i=1}^k b_i B_i\right) + \sum_{i=1}^k B_i^* Q B_i + \left(\tfrac{1}{2}\sum_{i=1}^k b_i^2 - 2\lambda\right) Q \leq 0. \tag{13}$$

Then (12) holds, and hence

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \leq \lambda, \quad \text{a.s.} \tag{14}$$

Proof: Note that for $i = 1, \ldots, k$, by the *abc*-formula,

$$\frac{\langle Qx, B_i x\rangle^2}{\langle Qx, x\rangle^2} + b_i \frac{\langle Qx, B_i x\rangle}{\langle Qx, x\rangle} + \frac{1}{4} b_i^2 \geq 0, \quad \text{for all } x \in \mathbb{R}^n. \tag{15}$$

So, by Proposition 3.1, if

$$2\langle QAx, x\rangle + \sum_{i=1}^k \langle QB_i x, B_i x\rangle - 2\lambda\langle Qx, x\rangle \leq -\sum_{i=1}^k 2b_i \langle Qx, B_i x\rangle - \frac{1}{2} b_i^2 \langle Qx, x\rangle, \quad \text{for all } x \in \mathbb{R}^n,$$

then the claimed result holds.

But this is equivalent to the stated condition. □

The following theorem gives sufficient conditions in order for a solution to (13) to exist.

3.6. **Theorem.** Suppose $L, D_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, k$ such that

$$\left|\left|e^{Lt}\right|\right| \leq m e^{\omega t} \quad \text{for all } t \geq 0,$$

with $m \geq 1$, $\omega \in \mathbb{R}$, and

$$m^2 \sum_{i=1}^k ||D_i||^2 + 2\omega < 0. \tag{16}$$

Then for any $M \in \mathbb{R}^{n\times n}$ there exists a unique solution $Q \in \mathbb{R}^{n\times n}$ to

$$L^* Q + QL + \sum_{i=1}^k D_i^* Q D_i = M. \tag{17}$$

This Q also satisfies

$$Q = \int_0^\infty e^{L^* t} \left(\sum_{i=1}^k D_i^* Q D_i - M\right) e^{Lt} \, dt. \tag{18}$$

Furthermore,

(i) if $M = 0$, then $Q = 0$,
(ii) if $M \leq 0$, then $Q \geq 0$, and
(iii) if $M < 0$ then $Q > 0$.

Proof: Define a recursion by

$$Q_0 := 0, \quad Q_{j+1} := \int_0^\infty e^{L^* t} \left(\sum_{i=1}^k D_i^* Q_j D_i - M\right) e^{Lt} \, dt.$$

The recursion is actually a contraction, since

$$\begin{aligned}||Q_{j+1}-Q_j|| &= \left|\left|\int_0^\infty \mathrm{e}^{L^*t}\left(\sum_{i=1}^k D_i^*(Q_j-Q_{j-1})D_i\right)\mathrm{e}^{Lt}\,dt\right|\right| \\ &\le m^2\sum_{i=1}^k ||D_i||^2\int_0^\infty \mathrm{e}^{2\omega t}\,dt\,||Q_j-Q_{j-1}|| \\ &= -\frac{m^2\sum_{i=1}^k ||D_i||^2}{2\omega}||Q_j-Q_{j-1}||.\end{aligned}$$

Note that the recursion is defined such that Q_{j+1} satisfies

$$L^*Q_{j+1}+Q_{j+1}L = M-\sum_{i=1}^k D_i^*Q_jD_i,$$

a basic result from Lyapunov theory (see e.g. [9]).

Hence there exists a unique fixed point $Q\in\mathbb{R}^{n\times n}$ that satisfies both (17) and

$$Q=\int_0^\infty \mathrm{e}^{L^*t}\left(\sum_{i=1}^k D_i^*QD_i-M\right)\mathrm{e}^{Lt}\,dt.$$

If $M=0$ then $Q=0$ by unicity of the solution.

Now suppose $M\le 0$. Then we can check that the recursion for (Q_j) has the property that $Q_j\ge 0$ for all j. So $Q\ge 0$, and (18) shows that

$$Q\ge -\int_0^\infty \mathrm{e}^{L^*t}M\mathrm{e}^{Lt}\,dt.$$

If $M<0$, then there exists a unique $P\in\mathbb{R}^{n\times n}$, $P>0$ such that $M=L^*P+PL$. Then

$$Q\ge -\int_0^\infty \mathrm{e}^{L^*t}M\mathrm{e}^{Lt}\,dt = P>0.$$

□

Consider now the case of only one noise term, that is equation (2). The following theorem is the main result on estimation of the Lyapunov exponent. It shows that based on estimates for $A+\sigma B$ and $B+\tau I$ for some constants $\sigma,\tau\in\mathbb{R}$, we can estimate the pathwise lyapunov exponent corresponding to (1).

3.7. **Theorem.** Suppose $h:\mathbb{R}\to[1,\infty)$, $g:\mathbb{R}\to\mathbb{R}$ and $f:\mathbb{R}\to[0,\infty)$ are functions such that

$$\begin{aligned}&||\exp\left((A+sB)t\right)||\le h(s)\exp(g(s)t)\quad\text{and}\\ &||B+sI||\le f(s),\quad s,t\in\mathbb{R}.\end{aligned}\tag{19}$$

Suppose for some $\sigma,\tau,\lambda\in\mathbb{R}$ we have

$$\lambda>\tfrac{1}{2}(h(\sigma-\tau)f(\tau))^2+g(\sigma-\tau)+\tfrac{1}{4}\sigma^2-\tfrac{1}{2}\tau^2.\tag{20}$$

Then

$$\lim_{t\to\infty}\frac{1}{t}\log|x(t)|\le\lambda\quad\text{a.s.},$$

where x is the solution to (2).

Proof: For the given combination of σ,τ and λ put

$$D:=B+\tau I,\quad L:=A+(\sigma-\tau)B+(\tfrac{1}{4}\sigma^2-\tfrac{1}{2}\tau^2-\lambda)I.$$

Note that

$$||D||\le f(\tau)\quad\text{and}\quad||\exp(Lt)||\le m\mathrm{e}^{\omega t},$$

with $m = h(\sigma-\tau)$ and $\omega = g(\sigma-\tau)+\frac{1}{4}\sigma^2-\frac{1}{2}\tau^2-\lambda$. Then by (20), condition (16) of Theorem 3.6 is satisfied and hence we can find a Q such that (17) holds for $M=-I$. By the choice of D and L, we have

$$0 \geq M = L^*Q + QL + D^*QD = (A+\sigma B)^*Q + Q(A+\sigma B) + B^*QB + (\tfrac{1}{2}\sigma^2 - 2\lambda)I,$$

and it follows that Q is a solution to (13), with $b=\sigma$.

Therefore by Lemma 3.5, we may conclude that (14) holds. □

3.8. **Example.** Consider the particular case where $B=\nu I$, and suppose ω is such that $||\exp(At)|| \leq e^{\omega t}$. Then

$$||\exp((A+sB)t)|| = ||\exp(At)||\exp(s\nu t) \leq e^{(\omega+s\nu)t},$$

and

$$||B+sI|| = |\nu+s|.$$

So

$$f(s) = |\nu+s|, \quad g(s) = \omega + s\nu, \quad \text{and} \quad h(s) = 1,$$

and we require that

$$\begin{aligned}\lambda &> \tfrac{1}{2}(\nu+\tau)^2 + \omega + (\sigma-\tau)\nu + \tfrac{1}{4}\sigma^2 - \tfrac{1}{2}\tau^2 \\ &= \tfrac{1}{2}\nu^2 + \omega + \sigma\nu + \tfrac{1}{4}\sigma^2,\end{aligned}$$

with minimizing $\sigma = -2\nu$, to obtain stability for any $\lambda > \omega - \frac{1}{2}\nu^2$. We see that in this example we recover the estimate of the commutative case, (11).

3.9. **Example.** Let

$$A = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}. \tag{21}$$

Equip $\mathbb{R}^2$ by the inner product $\langle x,y\rangle = x_1y_1 + qx_2y_2$, with $q>0$.

By Proposition 3.4, for all $s\in\mathbb{R}$ and $\varepsilon>0$ there exists a $q>0$ such that

$$\langle (A+sB)x,x\rangle \leq (\max(a_1+sb_1, a_2+sb_2)+\varepsilon)\langle x,x\rangle,$$

and therefore

$$\exp((A+sB)t) \leq \exp([\max(a_1+sb_1,a_2+sb_2)+\varepsilon]t), \quad t\geq 0.$$

Furthermore $||B+sI|| = \max(|b_1+s|,|b_2+s|)$, irrespective of q.

Hence conditions (19) hold with

$$\begin{aligned}&h(s)=1, \quad f(s) = \max(|b_1+s|,|b_2+s|), \quad \text{and} \\ &g(s) = \max(a_1+sb_1, a_2+sb_2)+\varepsilon.\end{aligned}$$

So, by Theorem 3.7, and letting ε by choosing ε arbitrarily small, if we pick

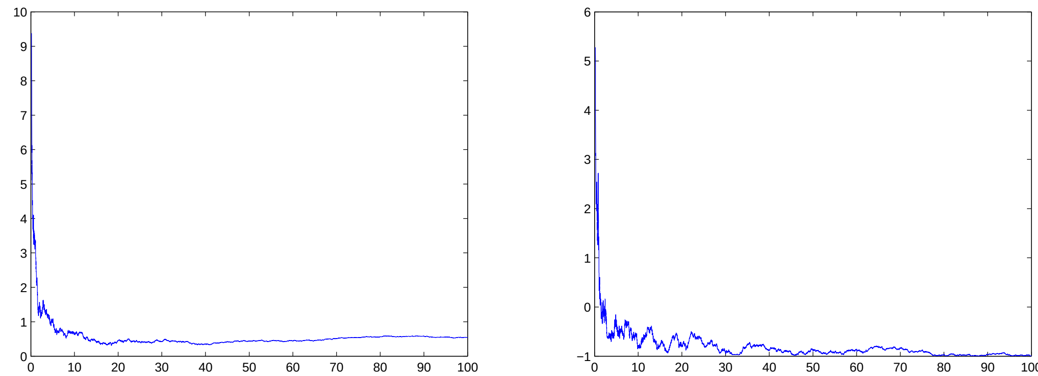
$$\begin{aligned}\lambda &> \tfrac{1}{2}\max((b_1+\tau)^2,(b_2+\tau)^2) + \max(a_1+(\sigma-\tau)b_1, a_2+(\sigma-\tau)b_2) + \tfrac{1}{4}\sigma^2 - \tfrac{1}{2}\tau^2 \\ &= \tfrac{1}{2}\max(2b_1\tau+b_1^2, 2b_2\tau+b_2^2) + \max(a_1+(\sigma-\tau)b_1, a_2+(\sigma-\tau)b_2) + \tfrac{1}{4}\sigma^2\end{aligned}$$

for some optimal σ and τ, then

$$\lim_{t\to\infty}\frac{1}{t}\log|x(t)| \leq \lambda,$$

when x is the solution of (2).

As a numerical example, let $a_1=1, a_2=-1, b_1=1$ and $b_2=0$. By picking $\tau=0$ and $\sigma=-2$, we see that any $\lambda>\frac{1}{2}$ is an upper bound for the Lyapunov exponent. If $a_1=1,a_2=-1,b_1=2$ and $b_2=0$, then by picking $\tau=-1$ and $\sigma=-2$ we obtain the estimate $\lambda>0$. We see that the noise has a stabilizing effect, even though it is degenerate. Now compare these theoretical results to a simulation (see Figure 3.9). We see that in the case $b_1=1$ the estimate is sharp, whereas in the case $b_1 = 2$ the graph suggest a Lyapunov exponent of -1, which gives room for even further theoretical improvements.



(a) $b_1 = 1$ (b) $b_1 = 2$

Figure 1: Graph of $\frac{1}{t} \log |x(t)|$ for a sample path of the solution x of (2) with A and B as given in (21), with $a_1 = 1$, $a_2 = -1$, $b_2 = 0$.

In a subsequent publication ([3]) the approach of this section is extended to infinite dimensional stochastic evolutions, with as application the case of stochastic differential equations with delay.

4. A necessary condition for pathwise stability

In this section we show that, in order for the solutions of the linear SDE (2) to be pathwise asymptotically stable, an assumption on the controllability properties of the pair (A, B) is necessary. First we introduce the necessary notions of stability.

4.1. **Definition.** The stochastic differential equation

$$\begin{cases} dx(t) = f(x(t), t) \ dt + g(x(t), t) \ dW(t), & t \geq 0, \\ x(0) = x_0 \end{cases}$$

is *pathwise asymptotically stable* if

$$\lim_{t \to \infty} |x(t; x_0)| = 0 \quad \text{almost surely,}$$

and *pathwise exponentially stable* if

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t; x_0)| < 0 \quad \text{almost surely}$$

for all initial conditions $x_0 \in \mathbb{R}^n$.

4.2. **Definition.** Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. The pair (A, B) is called *stochastically stabilizable* if there exists an $F \in \mathbb{R}^{m \times n}$ such that

$$dx(t) = Ax(t) \ dt + BFx(t) \ dW(t) \tag{22}$$

is pathwise asymptotically stable.

We would like to establish conditions on (A, B) such that (A, B) is stochastically stabilizable.

4.3. **Definition.** The pair (A, B) is called *(deterministically) stabilizable* if there exists an $F \in \mathbb{R}^{m\times n}$ such that $\mathfrak{s}(A + BF) < 0$. It is well known that, through pole placement, controllability implies stabilizability. Stabilizability can be understood as controllability of the unstable part of A.

We can now state a necessary condition for a system (A, B) to be stochastically stabilizable:

4.4. **Theorem.** Suppose (A, B) is stochastically stabilizable. Then (A, B) is stabilizable.

This leads directly to the following corollary which is the main result of this section.

4.5. **Corollary.** Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ such that the solution of (2) is asymptotically stable. Then (A, B) is stabilizable.

Proof (of corollary): By taking $F = I$, we see that (A, B) is stochastically stabilizable. Now apply the proposition. □

4.6. **Example.** In the commutative case of Section 2.2, if (A, B) satisfies $A - \frac{1}{2}B^2$ is stable, then (A, B) is stabilizable (by taking $F = -\frac{1}{2}B$). Hence the necessary condition for stochastic stability is satisfied, in agreement with Section 2.2.

In order to prove Theorem 4.4, we need some other notions and results from systems theory. See [13] for details.

4.7. **Controllability, isomorphic systems.** Recall that the pair (A, B) is called *controllable* if

$$\operatorname{rank}\left[B, AB, \ldots, A^{n-1}B\right] = n.$$

Here $[T_1, T_2, \ldots, T_n]$ denotes the concatenation of all the columns of matrices $T_1, \ldots, T_n$.

$\mu \in \mathbb{C}$ is called (A, B)-*controllable* if

$$\operatorname{rank}\left[A - \mu I, B\right] = n,$$

Note that if $\mu \notin \sigma(A)$, then μ is always (A, B)-controllable.

The system $(\overline{A}, \overline{B})$ is said to be *isomorphic* to (A, B) if there exists an invertible matrix S such that

$$\overline{A} = S^{-1}AS, \quad \overline{B} = S^{-1}B.$$

4.8. **Lemma.** If $\mu \in \sigma(A)$ is not (A, B)-controllable, then $\mu \in \sigma(A + BF)$ for all $F \in \mathbb{R}^{m\times n}$.

4.9. **Lemma.** Suppose (A, B) not controllable and $B \neq 0$. Then there exist $(\overline{A}, \overline{B})$ isomorphic to (A, B) such that

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

and such that (A_{11}, B_1) is controllable.

4.10. **Lemma.** Suppose (A, B) is not controllable and $(\overline{A}, \overline{B})$ is isomorphic to (A, B) such that

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

with (A_{11}, B_1) controllable. Then $\mu \in \mathbb{C}$ is (A, B)-controllable if and only if $\mu \notin \sigma(A_{22})$.

Proof of Theorem 4.4: Suppose (A, B) not stabilizable. Then by Lemma 4.8 there exists $\mu \in \sigma(A)$, $\Re\mu > 0$, such that μ is not (A, B)-controllable. By Lemma 4.9 and Lemma 4.10 there exist

$$\overline{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

isomorphic to (A, B) such that $\mu \in \sigma(A_{22})$. Now x satisfies

$$dx(t) = Ax(t)\,dt + BFx(t)\,dW(t), \tag{23}$$

if and only if $y(t) = Sx(t)$ satisfies

$$dy(t) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} y(t)\,dt + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \overline{F} y(t)\,dW(t), \tag{24}$$

where $\overline{F} = FS^{-1}$.

Let $y_2 \neq 0$ such that $A_{22}y_2 = \mu y_2$ and let $y_1 = 0$. Let y be the solution of (24) with initial condition $y(0) = [y_1\ y_2]^T$, and $x = S^{-1}y$ the corresponding solution of (23). Then

$$y_2(t) = \exp(A_{22}t)y_2 = \exp(\mu t)y_2, \quad \text{a.s.}$$

Hence

$$||S||\,|x(t)| \geq |Sx(t)| = |y(t)| \geq |y_2(t)| = \exp(\mu t)|y_2| \quad \text{a.s.}$$

and therefore

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \geq \Re\mu, \quad \text{a.s.}$$

Hence (A, B) is not stochastically stabilizable. □

Appendix A. Strong law of large numbers for martingales

Some of the proofs in this paper rely on the strong law of large numbers for martingales, which can be found in [11], Theorem 1.3.4, where it appears without proof. To make our exposition self-contained we provide the reader with a proof.

A.1. **Theorem (Strong law of large numbers for martingales).** Let $(M(t))_{t\geq 0}$ be a continuous local martingale with $M(0) = 0$. If

$$\limsup_{t\to\infty} \frac{[M](t)}{t} < \infty, \quad \text{a.s.}, \tag{25}$$

then

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0, \quad \text{a.s.}$$

Proof: Let $k, m \in \mathbb{N}$. For $n \in \mathbb{N}$, let E_n denote the event

$$E_n := \left\{ \sup_{t\geq 0} |M^{n+1}(t)| \geq \frac{n}{m} \text{ and } [M^{n+1}](\infty) \leq 2kn \right\}.$$

Here M^n denotes the martingale M stopped at time n.

Then by the exponential martingale inequality ([12], Exercise IV.3.16)

$$\mathbb{P}(E_n) \leq 2\mathrm{e}^{-\frac{n}{4km^2}}, \quad n \in \mathbb{N}.$$

Hence by Borel-Cantelli, $\mathbb{P}(E_n^c,\ \text{eventually}) = 1$. Let $\widetilde{\Omega}_{k,m} := (E_n^c,\ \text{eventually})$, and $\Omega_k := \{[M](t) \leq kt \text{ for all } t \geq 0\}$, for $k \in \mathbb{N}$.

On Ω_k, we have that

$$\frac{[M]^{n+1}(\infty)}{n} = \frac{[M](n+1)}{n} \leq \frac{(n+1)k}{n} \leq 2k, \quad n \in \mathbb{N},$$

so on $\Omega_k \cap \widetilde{\Omega}_{k,m}$ we have that

$$\sup_{t\geq 0} |M^{n+1}(t)| < \frac{n}{m}, \quad \text{eventually as } n \to \infty.$$

In particular, on $\Omega_k \cap \widetilde{\Omega}_{k,m}$, for $N \in \mathbb{N}$ large enough and $t \in [n, n+1]$, for $n > N$, $n \in \mathbb{N}$,

$$\frac{|M(t)|}{t} = \frac{|M^{n+1}(t)|}{t} \leq \frac{|M^{n+1}(t)|}{n} < \frac{1}{m},$$

that is

$$\frac{|M(t)|}{t} < \frac{1}{m} \quad \text{for } t \text{ large enough.}$$

Note that $\widetilde{\Omega}_k := \cap_m \widetilde{\Omega}_{k,m}$ has full measure and on $\Omega_k \cap \widetilde{\Omega}_k$, for all $m \in \mathbb{N}$ we have

$$\frac{|M(t)|}{t} < \frac{1}{m} \quad \text{for } t \text{ large enough.}$$

Note that, by (25), for all $\gamma > 0$ there exists an $k \in \mathbb{N}$ such that $\mathbb{P}(\Omega_k) \geq 1 - \gamma$. Therefore $\widetilde{\Omega} := \cup_k \widetilde{\Omega}_k$ has full measure and on $\widetilde{\Omega}$, for all $m \in \mathbb{N}$,

$$\frac{|M(t)|}{t} < \frac{1}{m} \quad \text{for } t \text{ large enough.}$$

□

Acknowledgement. The first and second author acknowledge the support by a 'VIDI subsidie' (639.032.510) of the Netherlands Organisation for Scientific Research (NWO).

We want to thank the anonymous referee for valuable comments and improvements.

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