Existence of an invariant measure for stochastic evolutions driven by an eventually compact semigroup

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Abstract. It is shown that for an SDE in a Hilbert space, eventual compactness of the driving semigroup together with compact perturbations can be used to establish the existence of an invariant measure.

The result is applied to stochastic functional differential equations and the heat equation perturbed by delay and noise, which are both shown to be driven by an eventually compact semigroup.

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1. Introduction

We consider here infinite-dimensional diffusions in a Hilbert space H described by the differential equation

$$\begin{cases} dX(t) = [AX(t) + F(X(t))]\ dt + G(X(t))\ dW(t), & t \geq 0, \\ X(0) = x, \end{cases} \tag{1.1}$$

with A the generator of a strongly continuous semigroup, F and G Lipschitz functions and W a Wiener process.

For many choices of A, F and G it is impossible to obtain the exact solution of such an equation. Therefore it is important to establish qualititative properties of the solution on the basis of information on A, F, G and W.

One of these qualitative properties is the existence of an invariant measure: under what conditions does a measure μ on H exist such that if the initial condition x has distribution μ, we have that $X(t)$ has distribution μ for all $t \geq 0$.

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Often a compactness argument (Krylov-Bogoliubov) is used to establish the existence of an invariant measure. In finite dimensions it suffices to show that the solutions of (1.1) are bounded in probability. In infinite dimensions, due to the absence of local compactness, we need to exploit compactness properties of the solutions of the stochastic differential equation.

It has been shown [3] that a suitable criterion is that A generates a compact semigroup. Together with solutions bounded in probability this suffices to prove the existence of an invariant measure. Another approach is taken in [9], based on hyperbolicity of the driving semigroup and small Lipschitz coefficients of the perturbations.

The result obtained by compactness of the semigroup leads immediately to the question whether eventual compactness of the semigroup can be used to establish existence of an invariant measure. This is an interesting question because, for example, delay differential equations, when put in an infinite-dimensional framework, possess this property (see [6]). Also in the theory of structured population equations eventually compact semigroups appear (see e.g. [7] and [8], Section VI.1). In [4] it is conjectured that eventual compactness should be a sufficient criterion for the existence of an invariant measure.

In this paper we show that eventual compactness of the semigroup, together with compact factorizations of the perturbations F and G, can indeed be used to establish the existence of an invariant measure (Section 2). As an example, the result is applied to a stochastic functional differential equation and the currently very active (see eg. [11]) field of reaction diffusion equations perturbed by delayed feedback and noise (Section 3). In Appendix A the eventual compactness of the delay semigroup, applied to partial differential equations, is established. This is a generalization of the well-known fact that ordinary delay differential equations are described by eventually compact semigroups.

2. Main result

Throughout this section, we will assume that the following hypothesis holds:

Hypothesis 2.1. (i) *H and U are separable Hilbert spaces and E_1 and E_2 are Banach spaces;*

(i) *A is the generator of a strongly continuous, eventually compact semigroup $(S(t))$ on H; we assume without loss of generality that $S(t)$ is compact for $t \geq 1$;*

(ii) *$F : H \to H$ is globally Lipschitz and admits a factorization $F = C_1 \circ \Phi$, where $C_1 \in L(E_1; H)$ is compact and $\Phi : H \to E_1$;*

(iii) *$G : H \to L_{HS}(U; H)$ is globally Lipschitz and admits a factorization $G(x) = C_2\Psi(x)$, $x \in H$, where $C_2 \in L(E_2; H)$ is compact, and $\Psi : H \to L_{HS}(U; E_2)$;* [1]

(iv) *W is a cylindrical Wiener process in U;*

[1]Here $L_{HS}(U; H)$ denotes the space of Hilbert-Schmidt operators from U into H.

(v) $(X(t,x))_{t\geq 0, x\in H}$ *is the unique mild solution ([4], Theorem 5.3.1) of the stochastic differential equation*

$$\begin{cases} dX(t) = (AX(t) + F(X))\ dt + G(X)\ dW(t), \\ X(0) = x, \quad x \in H \end{cases}$$

(vi) *For all* $x \in H$ *and* $\varepsilon > 0$, *there exists* $R > 0$ *such that for all* $T \geq 1$,

$$\frac{1}{T}\int_0^T \mathbb{P}(|X(t,x)| \geq R)\ dt < \varepsilon.$$

Under these assumptions, we will establish the existence of an invariant measure. First we need a couple of lemmas.

Lemma 2.2. *Let* E_1, E_2 *be Banach spaces. Let* $(T(t))$ *be a strongly continuous semigroup acting on* E_1 *and suppose* $C \in L(E_2; E_1)$ *is compact. Let*

$$Gf := \int_0^1 T(1-s)Cf(s)\ ds, \quad f \in L^p([0,1]; E_2),$$

with $p \geq 2$. *Then* $G \in L(L^p([0,1]; E_2); E_1)$ *is compact.*

Proof. Consider the set

$$V = \{T(t)Ck : t \in [0,1], k \in E_2, |k| \leq 1\}.$$

We will show that V is relatively compact. Indeed, let (v_n) be a sequence in V. There exist sequences $(t_n) \subset [0,1]$, $(x_n) \subset E_2$, with $|x_n| \leq 1$, $n \in \mathbb{N}$, such that

$$v_n = T(t_n)Cx_n, \quad n \in \mathbb{N}.$$

Since C is compact, there exists a subsequence (x_{n_k}) of (x_n) such that $Cx_{n_k} \to y$ for some $y \in E_1$, $|y| \leq ||C||$. Since $[0,1]$ is compact, by strong continuity of $(T(t))$, there exists a further subsequence $(t_{n_{k_l}})$ of (t_{n_k}) such that $T(t_{n_{k_l}})y \to z$ with $z \in T([0,1])y$.

Now

$$\begin{aligned} |T(t_{n_{k_l}})Cx_{n_{k_l}} - z| &\leq |T(t_{n_{k_l}})Cx_{n_{k_l}} - T(t_{n_{k_l}})y| + |T(t_{n_{k_l}})y - z| \\ &\leq ||T(t_{n_{k_l}})|||Cx_{n_{k_l}} - y| + |T(t_{n_{k_l}})y - z| \\ &\leq m|Cx_{n_{k_l}} - y| + |T(t_{n_{k_l}})y - z| \to 0 \end{aligned}$$

as $l \to \infty$. Here $m = \sup_{t\in[0,1]} ||T(t)||$.

So V is relatively compact and therefore its closed convex hull K is compact ([13], Theorem 3.25).

Now define a positive measure on $[0,1]$ by

$$\mu_f(ds) := |f(s)|\ ds.$$

Note that μ_f is a finite measure since, by Jensen,

$$\mu_f([0,1]) = \int_0^1 |f(s)|\ ds \leq \left(\int_0^1 |f(s)|^p\ ds\right)^{\frac{1}{p}}.$$

Now

$$Gf = \int_0^1 T(1-s)C \frac{f(s)}{|f(s)|} \ \mu_f(ds),$$

is an integral over positive, finite measure with the integrand assuming values in the convex set K, so

$$Gf \in \mu_f([0,1])K = ||f||_{L^p([0,1];E_2)}K.$$

□

We will need the following lemma.

Lemma 2.3. *Let H be a separable Hilbert space. Let $K \subset H$ be compact. Then there exists a compact, self-adjoint, strictly positive definite operator $T \in L(H)$ such that*

$$K \subset \{Tx : |x| \leq 1\}.$$

Proof. The proof is from [2], Example 3.8.13(ii). Assume for simplicity we deal with ℓ^2 and there exists a non-zero $x \in K$.

Claim.

$$\lim_{n\to\infty} \sup_{x \in K} \sum_{i=n}^{\infty} x_i^2 = 0.$$

Proof of claim. Suppose there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$, there exists $x^n \in K$ such that

$$\sum_{i=n}^{\infty} (x_i^n)^2 \geq \delta.$$

Now for fixed n pick such an x^n and $m \in \mathbb{N}$ such that

$$\sum_{i=m}^{\infty} (x_i^n)^2 < \delta/2.$$

Furthermore pick x^m such that

$$\sum_{i=m}^{\infty} (x_i^m)^2 \geq \delta.$$

Then

$$||x^n - x^m||_{\ell^2}^2 \geq ||(x^n - x^m)\mathbb{1}_{\{m,m+1,\ldots\}}||_{\ell^2}^2 > \delta/2.$$

So the sequence (x^n) does not have a Cauchy subsequence. Hence K is not compact, which proves the claim. ◇

So we can find an increasing sequence $(N_n)_{n=1}^{\infty}$ such that

$$\sum_{i=N_n}^{\infty} x_i^2 \leq 4^{-n} \quad \text{for all } x \in K.$$

Let $t_i > 0$, $t_i^2 := 2^{-n+1}$ for $N_n \leq i < N_{n+1}$, $n \in \mathbb{N}$, and $t_i^2 := 2 \sup_{x \in K} ||x||_{\ell^2}^2$ for $1 \leq i < N_1$. Define $T \in L(H)$ by $(Tx)_i := t_i x_i$.

Since $t_n \downarrow 0$, we see that T is compact. Furthermore, if $x \in K$, then let $y = (y_i)_{i=1}^\infty \in \ell^2$ with $y_i = \frac{x_i}{t_i}$, $i \in \mathbb{N}$. Then $Ty = x$, and

$$\sum_{i=1}^\infty y_i^2 = \sum_{i=1}^\infty \left(\frac{x_i}{t_i}\right)^2 = \sum_{i=1}^{N_1-1} \left(\frac{x_i}{t_i}\right)^2 + \sum_{n=1}^\infty \sum_{i=N_n}^{N_{n+1}-1} \left(\frac{x_i}{t_i}\right)^2$$

with

$$\sum_{i=N_n}^{N_{n+1}-1} \left(\frac{x_i}{t_i}\right)^2 = 2^{n-1} \sum_{i=N_n}^{N_{n+1}-1} x_i^2 \leq 2^{-n-1} \quad \text{and} \quad \sum_{i=1}^{N_1-1} \left(\frac{x_i}{t_i}\right)^2 \leq \frac{1}{2}.$$

We may conclude that

$$\sum_{i=1}^\infty y_i^2 \leq \frac{1}{2} + \sum_{n=1}^\infty 2^{-n-1} = 1,$$

so $y \in B(0,1)$. It follows that $K \subset T(B(0,1))$. □

Now consider, for $x \in H$, the stochastic variable

$$Y_x := \int_0^1 S(1-s)G(X(s,x))\, dW(s).$$

Lemma 2.4. *For all $\varepsilon > 0$ and $r > 0$, there exists a compact $K(r,\varepsilon) \subset H$ such that*

$$\mathbb{P}\left(Y_x \in K(r,\varepsilon)\right) > 1 - \varepsilon$$

for all $|x| \leq r$.

Proof. Recall the factorization $G = C_2\Psi$ through the Banach space E_2 from Hypothesis 2.1 with C_2 compact. In the proof of Lemma 2.2, it is shown that if we let

$$V = \left\{S(t)C_2k : t \in [0,1], k \in E_2, |k| \leq 1\right\},$$

and K the closed convex hull of V, then K is compact. Let $T \in L(H)$, compact, be as given by Lemma 2.3, so $K \subset T(B(0,1))$ and since T is injective, $V \subset K \subset \mathfrak{D}(T^{-1})$.

Let $K(\lambda) := \lambda T(B(0,1))$ for $\lambda > 0$, where $B(0,1)$ is the unit ball in H.

Note that

$$\begin{aligned} Y_x &= \int_0^1 S(1-s)C\Psi(X(s,x))\, dW(s) = \int_0^1 TT^{-1}S(1-s)C\Psi(X(s,x))\, dW(s) \\ &= T\int_0^1 T^{-1}S(1-s)C\Psi(X(s,x))\, dW(s). \end{aligned}$$

So

$$Y_x \in K(\lambda) \Leftrightarrow \int_0^1 T^{-1}S(1-s)C\Psi(X(s,x))\, dW(s) \in \lambda B(0,1).$$

Hence, using the fact that $T^{-1}S(1-s)C$ is an operator of norm not greater than 1 (by definition of T),

$$\begin{aligned}\mathbb{P}(Y_x \notin K(\lambda)) &\leq \mathbb{P}\left(\int_0^1 T^{-1}S(1-s)C\Psi(X(s,x))\,dW(s) \notin \lambda B(0,1)\right)\\ &\leq \frac{1}{\lambda^2}\mathbb{E}\left[\left|\int_0^1 T^{-1}S(1-s)C\Psi(X(s,x))\,dW(s)\right|^2\right]\\ &= \frac{1}{\lambda^2}\mathbb{E}\left[\int_0^1 \left|T^{-1}S(1-s)C\Psi(X(s,x))\right|_{HS}^2\,ds\right]\\ &\leq \frac{c_1}{\lambda^2}\mathbb{E}\left[\int_0^1 |\Psi(X(s,x))|_{HS}^2\,ds\right] \leq \frac{c_2}{\lambda^2}(1+|x|^2),\end{aligned}$$

for some constants $c_1, c_2 > 0$, and where we used [4], Theorem 5.3.1 in the last step. Now pick λ large enough such that

$$\frac{c_2}{\lambda^2}(1+r^2) < \varepsilon.$$

□

Lemma 2.5. *Suppose Hypothesis* 2.1 *is satisfied. For any* $\varepsilon > 0$ *and* $r > 0$ *there exists a compact* $K(r,\varepsilon) \subset H$ *such that*

$$\mathbb{P}(X(1,x) \in K(r,\varepsilon)) \geq 1-\varepsilon \quad \textit{for all } x \in H \textit{ with } |x| \leq r.$$

Proof. Note that

$$X(1,x) = S(1)x + \int_0^1 S(1-s)F(X(s,x))\,ds + \int_0^1 S(1-s)G(X(s,x))\,dW(s).$$

We treat the three terms separately.

Since $S(1)$ is a compact operator, for any $r > 0$ there exists a compact set $K_1(r)$ such that $S(1)x \in K_1(r)$ for all $|x| \leq r$.

Let $p \geq 2$. From [4], Theorem 5.3.1, it follows that there exists a constant $k > 0$ such that

$$\mathbb{E}\left[\int_0^1 |\Phi(X(s,x))|^p\,ds\right] \leq k(1+|x|^p).$$

Define

$$f : \Omega \times [0,1] \to E_1, \quad f(t) := \Phi(X(t,x)), \quad t \in [0,1].$$

Then for $\lambda > 0$,

$$\mathbb{P}(|f|_{L^p(0,1;E_1)} > \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}\left[|f|_{L^p(0,1;E_1)}^p\right] \leq \frac{k}{\lambda^p}(1+|x|^p) \leq \frac{k}{\lambda^p}(1+r^p).$$

Pick

$$\lambda := \left(\frac{2k}{\varepsilon}(1+r^p)\right)^{1/p},$$

so that

$$\mathbb{P}(|f|_{L^p(0,1;E_1)} > \lambda) \leq \varepsilon/2.$$

Hence, by Lemma 2.2 there exists a compact set $K_2(\lambda) = K_2(r, \varepsilon)$ such that

$$\mathbb{P}\left(\int_0^1 S(1-s)F(X(s,x))\, ds \in K_2(r,\varepsilon) \right) > 1 - \varepsilon/2.$$

By Lemma 2.4, there exists a compact set $K_3(r, \varepsilon)$ such that

$$\mathbb{P}\left(\int_0^1 S(1-s)G(X(s,x))\, ds \in K_3(r,\varepsilon) \right) > 1 - \varepsilon/2.$$

We may conclude that

$$\mathbb{P}(X(1,x) \in K_1(r) + K_2(r,\varepsilon) + K_3(r,\varepsilon)) \geq 1 - \varepsilon.$$

□

Theorem 2.6. *Suppose Hypothesis* 2.1 *is satisfied. Then there exists an invariant measure for* $(X(t,x))_{t \geq 0}$.

Proof. The proof is analogous to the proof of [4], Theorem 6.1.2.

Let $K(r, \varepsilon)$ as in Lemma 2.5. For $t > 1$, using Markov transition probabilities $(p_t(x,\ dy))$,

$$\begin{aligned} &\mathbb{P}(X(t,x) \in K(r,\varepsilon)) = \mathbb{E}\left[p_1(X(t-1,x), K(r,\varepsilon))\right] \\ &\geq \mathbb{E}\left[p_1(X(t-1,x), K(r,\varepsilon)) \mathbb{1}_{\{|X(t-1,x)| \leq r\}}\right]. \end{aligned}$$

By Lemma 2.5,

$$\mathbb{P}(X(t,x) \in K(r,\varepsilon)) \geq (1-\varepsilon)\mathbb{P}(|X(t-1,x)| \leq r),$$

so

$$\frac{1}{T}\int_1^{T+1} \mathbb{P}(X(t,x) \in K(r,\varepsilon))\, dt \geq \frac{1-\varepsilon}{T}\int_0^T \mathbb{P}(|X(t,x)| \leq r)\, dt.$$

Now, using Hypothesis 2.1, (vii), take r large enough and ε small enough, to see that the family

$$\frac{1}{T}\int_1^{T+1} p_t(x,\cdot)\, dt, \quad T \geq 1,$$

is tight. By Krylov-Bogoliubov there exists an invariant measure. □

It is currently not known to us whether an invariant measure always exists if F and G are only Lipschitz without a factorization property as in Hypothesis 2.1 (ii) and (iii).

3. Example: stochastic evolutions with delay

Evolutions with delayed dependence have been studied for some time now. One can think of both ordinary differential equations and partial differential equations, perturbed by a dependence on the 'past' of the process, leading to functional differential equations (see [5], [6], [10]) and partial functional differential equations (see [14]), respectively. The latter class has attracted a lot of research activity recently, see for example [11].

Can we establish the existence of an invariant measure for such evolutions, perturbed by noise? In order to answer this question, we first present the abstract framework in the style of [1].

Let X, Z be a Banach spaces. Consider the abstract differential equation with delay

$$\begin{cases} \frac{d}{dt}u(t) = Bu(t) + \Phi u_t, \quad t > 0, \\ u(0) = x, \\ u_0 = f, \end{cases} \tag{3.1}$$

where

(i) $x \in X$;
(ii) B the generator of a strongly continuous semigroup $(S(t))$ in X;
(iii) $\mathfrak{D}(B) \overset{d}{\hookrightarrow} Z \overset{d}{\hookrightarrow} X$;[2]
(iv) $f \in L^p([-1,0]; Z)$, $1 \leq p < \infty$;
(v) $\Phi : W^{1,p}([-1,0]; Z) \to X$ a bounded linear operator[3];
(vi) $u : [-1,\infty) \to X$ and for $t \geq 0$, $u_t : [-1,0] \to X$ is defined by $u_t(\sigma) = u(t+\sigma)$, $\sigma \in [-1,0]$.

A *classical solution* of (3.1) is a function $u : [-1,\infty) \to X$ that satisfies

(i) $u \in C([-1,\infty); X) \cap C^1([0,\infty); X)$;
(ii) $u(t) \in \mathfrak{D}(B)$ and $u_t \in W^{1,p}([-1,0]; Z)$ for all $t \geq 0$;
(iii) u satisfies (3.1) for all $t \geq 0$.

To employ a semigroup approach we introduce the Banach space

$$\mathcal{E}^p := X \times L^p([-1,0]; Z),$$

and the closed, densely defined operator in $\mathcal{E}^p$,

$$A := \begin{bmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{bmatrix}, \quad \mathfrak{D}(A) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in X \times W^{1,p}([-1,0]; Z) : f(0) = x \right\}. \tag{3.2}$$

The equation (3.1) is called *wellposed* if for all $(x, f) \in \mathfrak{D}(A)$, there exists a unique classical solution of (3.1) that depends continuously on the initial data (in the sense of uniform convergence on compact intervals).

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[2]Here $\overset{d}{\hookrightarrow}$ denotes continuous and dense embedding
[3]Here $W^{k,p}(U; V)$ denotes the Sobolev space consisting of equivalence classes of functions mapping from U into V with partial derivatives up to and including k-th order in $L^p(U; V)$

It is shown in [1] that A generates a strongly continuous semigroup in $\mathcal{E}^p$ if and only if (3.1) is wellposed. Furthermore, sufficient conditions on Φ are given for this to be the case:

Hypothesis 3.1. *Let $S_t : X \to L^p([-1,0];Z)$ be defined by*

$$(S_t x)(\tau) := \begin{cases} S(t+\tau)x & \text{if } -t < \tau \le 0, \\ 0 & \text{if } -1 \le \tau \le -t, \end{cases} \qquad t \ge 0.$$

Let $(T_0(t))_{t\ge 0}$ be the nilpotent left shift semigroup on $L^p([-1,0];Z)$. Assume that there exists a function $q : [0,\infty) \to [0,\infty)$ with $\lim_{t\downarrow 0} q(t) = 0$, such that

$$\int_0^t ||\Phi(S_r x + T_0(r) f)||\, dr \le q(t) \left|\left| \begin{pmatrix} x \\ f \end{pmatrix} \right|\right|$$

for all $t > 0$ and $\begin{pmatrix} x \\ f \end{pmatrix} \in \mathfrak{D}(A)$. Furthermore suppose that either

(A) $Z = X$ *or*
(B) (i) $(B, \mathfrak{D}(B))$ *generates an analytic semigroup $(S(t))_{t\ge 0}$ on X, and*
(ii) *for some $\delta > \omega_0(B)$ there exists $\vartheta < \frac{1}{p}$ such that*

$$\mathfrak{D}((-B+\delta I)^{\vartheta}) \overset{d}{\hookrightarrow} Z \overset{d}{\hookrightarrow} X.$$

Theorem 3.2. *Assume Hypothesis* 3.1 *holds. Then $(A, \mathfrak{D}(A))$ is the generator of a strongly continuous semigroup $(T(t))_{t\ge 0}$ on $\mathcal{E}^p$.*

Proof. See [1], Theorem 3.26 and Theorem 3.34. □

Example. Let $\Phi : C([-1,0];Z) \to X$ be given by

$$\Phi(f) := \int_{-1}^{0} d\eta\ f,$$

where $\eta : [-1,0] \to L(Z;X)$ is of bounded variation.

Suppose either that $Z = X$ or that $(B, \mathfrak{D}(B))$ satisfies the assumptions (B-i) and (B-ii) of Hypothesis 3.1. Then the conditions of Theorem 3.2 are satisfied and hence $(A, \mathfrak{D}(A))$ generates a strongly continuous semigroup (see [1], Theorem 3.29 and Theorem 3.35). ◇

By the following theorem, proven in Section A, we see that in many cases A generates an eventually compact semigroup.

Theorem 3.3. *Suppose Hypothesis* 3.1 *holds. Furthermore suppose $(S(t))_{t\ge 0}$ is immediately compact. Then $(T(t))_{t\ge 0}$ is compact for all $t > 1$. If X is finite dimensional, then $(T(t))_{t\ge 0}$ is also compact at $t = 1$.*

3.1. Example (Functional differential equations)

A relatively easy case is now the example of a functional differential equation perturbed by noise. In the framework introduced above, let $X = Z = \mathbb{R}^d$, $B \in L(\mathbb{R}^d)$ and $\Phi(f) = \int_{-1}^0 d\eta\, f$, with $\eta : [-1,0] \to L(\mathbb{R}^d)$ of bounded variation.

As a special case, we can take $\eta(\sigma) = \sum_{i=1}^n H(\sigma - \theta_i) B_i$, where H denotes the Heaviside step function, $\theta_i \in [-1,0]$, and $B_i \in L(\mathbb{R}^d)$, $i = 1, \dots n$. Then (3.1) becomes the delay differential equation

$$\frac{du}{dt} = Bu(t) + \sum_{i=1}^n B_i u(t - \theta_i).$$

We can perturb the functional differential equation by noise to obtain a stochastic functional differential equation of the form

$$du = \left[Bu(t) + \int_{-1}^0 d\eta\, u_t + \varphi(u(t), u_t) \right] dt + \psi(u(t), u_t)\, dW(t), \quad t \geq 0, \qquad (3.3)$$

where $\varphi : \mathcal{E}^2 \to \mathbb{R}^d$, $\psi : \mathcal{E}^2 \to L(\mathbb{R}^m; \mathbb{R}^d)$ are Lipschitz, and $(W(t))_{t \geq 0}$ is an m-dimensional standard Brownian motion.

If we define $F : \mathcal{E}^2 \to \mathcal{E}^2$ and $G : \mathcal{E}^2 \to L(\mathbb{R}^m; \mathcal{E}^2)$ by

$$F\left(\begin{bmatrix} v \\ w \end{bmatrix} \right) := \begin{bmatrix} \varphi(v, w) \\ 0 \end{bmatrix}, \quad G\left(\begin{bmatrix} v \\ w \end{bmatrix} \right) := \begin{bmatrix} \psi(v, w) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} v \\ w \end{bmatrix} \in \mathcal{E}^2,$$

and A as in (3.2), then we arrive in the framework of equation (1.1) with as state space $H = \mathcal{E}^2$.

Since F and G map into finite dimensional subspaces of $\mathcal{E}^2$, they clearly admit a compact factorization as meant in Hypothesis 2.1. By Theorem 3.3, A generates an eventually compact semigroup. Hence all conditions of Hypothesis 2.1 are satisfied, except possibly condition (vi), boundedness in probability.[4] If this condition is also satisfied, by Theorem 2.6 we have established the existence of an invariant measure for (3.3) on the state space $\mathcal{E}^2$.

3.2. Example

Reaction diffusion equations with delayed nonlocal reaction terms are a topic of active research in the study of biological invasion and disease spread. Can we establish the existence of an invariant measure if we add randomness to such a system? As an example we set out to answer this question for an equation similar to one encountered in e.g. [11].

[4] Boundedness in probability has to be established for individual cases, for example, by posing dissipativity conditions on A, F and G, see [3].

Consider the reaction diffusion equation with delay and noise

$$\begin{cases} du(t,\xi) = \Big[\Delta_\xi u(t,\xi) + \sum_{i=1}^n c_i \frac{\partial}{\partial\xi} u(t-\theta_i,\xi) + \varphi(u_t)(\xi)\Big]\ dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + (\sigma \circ dW)(t,\xi), & t \geq 0, \xi \in \mathbb{R}, \\ \lim_{\xi\to\pm\infty} u(t,\xi) = \lim_{\xi\to\pm\infty} \frac{\partial u}{\partial\xi}(t,\xi) = 0, & t \geq 0, \\ u(t,\xi) = f(t,\xi), & t \in [-1,0], \xi \in \mathbb{R}, \\ u(0,\xi) = v(\xi), & \xi \in \mathbb{R}. \end{cases} \tag{3.4}$$

with

(i) delay parameters $c_i \in \mathbb{R}, \theta_i \in [-1,0]$, $i = 1,\ldots,n$,
(ii) initial conditions $f \in L^2([-1,0];W^{1,2}(\mathbb{R}))$ and $v \in L^2(\mathbb{R})$,
(iii) Lipschitz reaction term $\varphi : L^2([-1,0];W^{1,2}(\mathbb{R})) \to L^2(\mathbb{R})$ (possibly non-linear and/or non-local),
(iv) $u_t \in L^2([-1,0];W^{1,2}(\mathbb{R}))$ denoting the segment process defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-1,0]$, $t \geq 0$,
(v) $(W(t))_{t\geq 0}$ a cylindrical Brownian motion in some Hilbert space U,
(vi) noise factor $\sigma \in L_{HS}(U;W^{1,2}(\mathbb{R}))$.

We can employ the semigroup approach discussed before by setting $X := L^2(\mathbb{R})$, $Z := W^{1,2}(\mathbb{R})$, as state space the Hilbert space $\mathcal{E}^2 = X \times L^2([-1,0];Z)$, with A as defined in (3.2), with

$$B := \Delta, \quad \mathfrak{D}(B) = \left\{ v \in W^{2,2}(\mathbb{R}) : \lim_{\xi\to\pm\infty} v(\xi) = 0, \lim_{\xi\to\pm\infty} \frac{\partial v}{\partial\xi}(\xi) = 0 \right\}$$

and

$$\Phi(w) := \sum_{i=1}^n c_i \frac{\partial}{\partial\xi} w(t-\theta_i,\xi), \quad w \in L^2([-1,0];W^{1,2}(\mathbb{R})).$$

Then A is of the form described in Example 3. Since B generates an immediately compact semigroup (see for example [8], Exercise II.4.30(4)), it follows from Theorem 3.3 that A generates an eventually compact semigroup.

Furthermore let

$$F\left(\begin{bmatrix} v \\ w \end{bmatrix}\right) := \begin{bmatrix} \varphi(w) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} v \\ w \end{bmatrix} \in \mathcal{E}^2, \quad \text{and} \quad G(\cdot) := \begin{bmatrix} \sigma \\ 0 \end{bmatrix} \quad \text{on } \mathcal{E}^2.$$

Then (3.4) is described by (1.1) in the state space $H = \mathcal{E}^2$.

It remains to impose conditions on the nonlinear term F. Let us require, for example, that $\varphi : L^2([-1,0];W^{1,2}(\mathbb{R})) \to L^2(\mathbb{R})$ is of the form

$$\varphi(w) := (g \circ h)(w),$$

with $g : W^{1,2}(\mathbb{R}) \to W^{1,2}(\mathbb{R})$ (possibly the identity mapping), and h defined by

$$h(w)(\xi) := \int_{-1}^0 \int_{\mathbb{R}} k(\eta,\sigma)\psi(w(\sigma,\xi-\eta))\ d\eta\ d\sigma, \quad \xi \in \mathbb{R},$$

where $\psi \in W^{1,\infty}(\mathbb{R})$ with $|\psi(\zeta)| \leq ||\dot{\psi}||_\infty |\zeta|$, $\zeta \in \mathbb{R}$, and $k \in L^1(\mathbb{R};L^2([-1,0]))$.

We will now verify that in this case

$$\varphi : L^2([-1,0]; W^{1,2}(\mathbb{R})) \to W^{1,2}(\mathbb{R}). \tag{3.5}$$

Indeed, using Fubini, Cauchy-Schwarz and Young's inequality for convolutions,

$$\begin{aligned}
\int_{\mathbb{R}} |h(w)(\xi)|^2 \, d\xi &= \int_{\mathbb{R}} \left| \int_{-1}^0 \int_{\mathbb{R}} k(\eta,\sigma)\psi(w(\sigma,\xi-\eta)) \, d\eta \, d\sigma \right|^2 \, d\xi \\
&\leq ||\dot\psi||_\infty^2 \int_{\mathbb{R}} \left(\int_{-1}^0 \int_{\mathbb{R}} |k(\eta,\sigma) w(\sigma,\xi-\eta)| \, d\eta \, d\sigma \right)^2 \, d\xi \\
&\leq ||\dot\psi||_\infty^2 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} ||k(\eta,\cdot)||_{L^2([-1,0])} ||w(\cdot,\xi-\eta)||_{L^2([-1,0])} \, d\eta \right)^2 \, d\xi \\
&\leq \left(||\dot\psi||_\infty ||k||_{L^1(\mathbb{R};L^2([-1,0]))} \; ||w||_{L^2([-1,0];L^2(\mathbb{R}))} \right)^2 .
\end{aligned}$$

Furthermore, using the same classic inequalities,

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \frac{\partial}{\partial \xi} h(w)(\xi) \right|^2 \, d\xi \\
&= \int_{\mathbb{R}} \left| \int_{-1}^0 \int_{\mathbb{R}} k(\eta,\sigma) \frac{\partial}{\partial \xi} \psi(w(\sigma,\xi-\eta)) \, d\eta \, d\sigma \right|^2 \, d\xi \\
&= \int_{\mathbb{R}} \left| \int_{-1}^0 \int_{\mathbb{R}} k(\eta,\sigma) \dot\psi(w(\sigma,\xi-\eta)) \frac{\partial}{\partial \xi} w(\sigma,\xi-\eta) \, d\eta \, d\sigma \right|^2 \, d\xi \\
&\leq ||\dot\psi||_\infty^2 \int_{\mathbb{R}} \left(\int_{-1}^0 \int_{\mathbb{R}} \left| k(\eta,\sigma) \frac{\partial}{\partial \xi} w(\sigma,\xi-\eta) \right| \, d\eta \, d\sigma \right)^2 \, d\xi \\
&\leq ||\dot\psi||_\infty^2 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} ||k(\eta,\cdot)||_{L^2([-1,0])} \left|\left| \frac{\partial}{\partial \xi} w(\cdot,\xi-\eta) \right|\right|_{L^2([-1,0])} \, d\eta \right)^2 \, d\xi \\
&\leq \left(||\dot\psi||_\infty ||k||_{L^1(\mathbb{R};L^2([-1,0]))} \; ||w||_{L^2([-1,0];W^{1,2}(\mathbb{R}))} \right)^2 .
\end{aligned}$$

So we have $h : L^2([-1,0]; L^2(\mathbb{R})) \to W^{1,2}(\mathbb{R})$, and therefore (3.5) holds for $\varphi = g \circ h$. Hence in this case we may write (with some abuse of notation) $\varphi = \imath \circ \varphi$, where $\imath : W^{1,2}(\mathbb{R}) \to L^2(\mathbb{R})$ is the canonical embedding of $W^{1,2}(\mathbb{R})$ into $L^2(\mathbb{R})$, which is a compact mapping. We conclude that F admits a compact factorization. Note that this carries over to any function φ that satisfies (3.5). G admits a compact factorization as well, again using the compact embedding of $W^{1,2}(\mathbb{R})$ into $L^2(\mathbb{R})$.

Again, we may conclude from Theorem 2.6 that if the solutions of (3.4) are bounded in probability, an invariant measure exists.

Appendix A. Eventual compactness of the delay semigroup

The purpose of this section is to prove Theorem 3.3. We proceed as in [8], Section VI.6. We will use the following variant of the Arzelà-Ascoli theorem.

Definition A.1. A subset Φ of $C(X;Y)$ is *pointwise relatively compact* if and only if $\forall x \in X$, the set $\{f(x) : f \in \Phi\}$ is relatively compact in Y.

Theorem A.2 (vector valued Arzelà-Ascoli, [12]**, Theorem 47.1).** *Let X be a compact Hausdorff space and Y a metric space. Then a subset Φ of $C(X;Y)$ is relatively compact if and only if it is equicontinuous and pointwise relatively compact.*

Lemma A.3. *Suppose $(S(t))_{t\geq 0}$ is immediately compact. Then $R(\lambda, A)T(1)$ is compact for all $\lambda \in \rho(A)$.*

Proof. According to [1], Proposition 3.19, we have the following expression for the resolvent $R(\lambda, A)$:

$$R(\lambda, A) = \begin{bmatrix} R(\lambda, B + \Phi_\lambda) & R(\lambda, B + \Phi_\lambda)\Phi R(\lambda, A_0) \\ \epsilon_\lambda R(\lambda, B + \Phi_\lambda) & [\epsilon_\lambda R(\lambda, B + \Phi_\lambda)\Phi + I]R(\lambda, A_0) \end{bmatrix}, \quad \lambda \in \rho(A), \tag{A.1}$$

where, for $\lambda \in \mathbb{C}$, $\Phi_\lambda \in L(X)$ is given by

$$\Phi_\lambda x := \Phi(e^{\lambda \cdot} x), \quad x \in X,$$

ϵ_λ is the function

$$\epsilon_\lambda(s) := \mathrm{e}^{\lambda s}, \quad s \in [-1, 0].$$

and A_0 is the generator of the nilpotent left-shift semigroup on $L^p([-1, 0]; X)$.

Let

$$\pi_1 : X \times L^p([-1, 0]; Z) \to X$$

and

$$\pi_2 : X \times L^p([-1, 0]; Z) \to L^p([-1, 0]; Z)$$

denote the canonical projections on X and $L^p([-1, 0]; Z)$, respectively.

Lemma 4.5 and Lemma 4.9 in [1] state that the operator $R(\lambda, B + \Phi_\lambda)$ is compact for all $\lambda \in \rho(A)$. Therefore, using (A.1)

$$\pi_1 R(\lambda, A)T(1) = \begin{bmatrix} R(\lambda, B + \Phi_\lambda) & R(\lambda, B + \Phi_\lambda)\Phi R(\lambda, A_0) \end{bmatrix} T(1) \tag{A.2}$$

is compact.

We can therefore restrict our attention to

$$\pi_2 R(\lambda, A)T(1) : X \times L^p([-1, 0]; Z) \to L^p([-1, 0]; Z).$$

Denote $\varphi := \begin{pmatrix} x \\ f \end{pmatrix}$ where $x \in X$ and $f \in L^p([-1, 0]; Z)$. Note that

$$\frac{d}{d\sigma}\pi_2 R(\lambda, A)T(1)\varphi = \pi_2 A R(\lambda, A)T(1)\varphi,$$

Hence, using Hölder, there exists some constant $M > 0$ such that for all $t_0, t_1 \in [-1, 0]$,

$$\begin{aligned}
&||\pi_2 R(\lambda, A)T(1)\varphi(t_1) - \pi_2 R(\lambda, A)T(1)\varphi(t_0)||_Z \\
&= \left|\left|\int_{t_0}^{t_1} \left[\frac{d}{d\sigma}\pi_2 R(\lambda, A)T(1)\varphi\right](\sigma)\ d\sigma\right|\right|_Z \\
&= \left|\left|\int_{t_0}^{t_1} \left[\pi_2 AR(\lambda, A)T(1)\varphi\right](\sigma)\ d\sigma\right|\right|_Z \\
&\leq \int_{t_0}^{t_1} ||[\pi_2 AR(\lambda, A)T(1)\varphi](\sigma)||_Z\ d\sigma \\
&\leq |t_1 - t_0|^{1/q}\, ||\pi_2 AR(\lambda, A)T(1)\varphi||_{L^p([-1,0];Z)} \\
&\leq M|t_1 - t_0|^{1/q}\, ||\varphi||_{\mathcal{E}^p}.
\end{aligned}$$

Here $q > 1$ is such that $\frac{1}{q} + \frac{1}{p} = 1$.

So

$$\mathcal{C} := \{\pi_2 R(\lambda, A)T(1)\varphi : \varphi \in \mathcal{E}^p, ||\varphi||_{\mathcal{E}^p} \leq 1\} \subset C([-1, 0]; Z)$$

is equicontinuous.

Furthermore note that

$$\begin{aligned}
&[\pi_2 R(\lambda, A)T(1)\varphi](\sigma) && \\
&= [\pi_2 T(1)R(\lambda, A)\varphi](\sigma) && \text{(commutativity of } R(\lambda, A) \text{ and } T(1)) \\
&= [\pi_2 T(1+\sigma)R(\lambda, A)\varphi](0) && \text{(translation property of } (T(t))_{t\geq 0}) \\
&= [\pi_2 R(\lambda, A)T(1+\sigma)\varphi](0) && \text{(commutativity)} \\
&= \pi_1 R(\lambda, A)T(1+\sigma)\varphi && (R(\lambda, A) \text{ maps to domain } A).
\end{aligned}$$

Again using (A.1) and the fact that $R(\lambda, B + \Phi_\lambda)$ is compact, we find that $\mathcal{C}$ is pointwise relatively compact. By the vector-valued Arzelà-Ascoli theorem, Theorem A.2, we find that $\mathcal{C}$ is relatively compact in $C([-1, 0]; Z)$ and hence relatively compact in $L^p([-1, 0]; Z)$.

From this we conclude that $\pi_2 R(\lambda, A)T(1)$ is compact and combining this with (A.2), $R(\lambda, A)T(1)$ is compact. □

We may now conclude that $(T(t))_{t\geq 0}$ is eventually compact:

Theorem A.4. *Suppose Hypothesis* 3.1 *holds. Furthermore suppose* $(S(t))_{t\geq 0}$ *is immediately compact. Then* $(T(t))_{t\geq 0}$ *is compact for all* $t > 1$. *If* X *is finite dimensional, then* $(T(t))_{t\geq 0}$ *is also compact at* $t = 1$.

Proof. By [8], Lemma II.4.28, it is sufficient to show that $(T(t))$ is eventually norm continuous for $t > 1$, and that $R(\lambda, A)T(1)$ is compact for some $\lambda \in \rho(A)$.

Now by [1], Lemma 4.5, $(T(t))$ is norm continuous for $t > 1$ (using that $(S(t))$ is immediately compact and hence immediately norm continuous). Furthermore by Lemma A.3, $R(\lambda, A)T(1)$ is compact for all $\lambda \in \rho(A)$.

For the finite dimensional case, see [6]. □

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